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Regularity of solutions to fully nonlinear elliptic and parabolic free boundary problems [☆]

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Abstract

We consider fully nonlinear obstacle-type problems of the form

$$\begin{cases} F(D^2u, x) = f(x) & \text{a.e. in } B_1 \cap \Omega, \\ |D^2u| \le K & \text{a.e. in } B_1 \backslash \Omega, \end{cases}$$

where Ω is an open set and $K > 0$. In particular, structural conditions on F are presented which ensure that $W^{2,n}(B_1)$ solutions achieve the optimal $C^{1,1}(B_{1/2})$ regularity when f is Hölder continuous. Moreover, if f is positive on $\overline{B}_1$, Lipschitz continuous, and $\{u \neq 0\} \subset \Omega$, we obtain interior C^1 regularity of the free boundary under a uniform thickness assumption on $\{u = 0\}$. Lastly, we extend these results to the parabolic setting.

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1. Introduction

Obstacle-type problems appear in several mathematical disciplines such as minimal surface theory, potential theory, mean field theory of superconducting vortices, optimal control, fluid filtration in porous media, elasto-plasticity, and financial mathematics [1–5]. The classical obstacle problem involves minimizing the Dirichlet energy on a given domain in the space of square integrable functions with square integrable gradient constrained to remain above a fixed

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obstacle function and with prescribed boundary data. Due to the structure of the Dirichlet integral, this minimization process leads to the free boundary problem

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in } B_1,$$

where $B_1 \subset \mathbb{R}^n$ is the unit ball centered at the origin. A simple one-dimensional example shows that even if $f \in C^\infty$, u is not more regular than $C^{1,1}$, and Lipschitz continuity of f yields this optimal regularity via the Harnack inequality.

An obstacle-type problem is a free boundary problem of the form

$$\Delta u = f\chi_\Omega \quad \text{in } B_1, \tag{1}$$

where Ω is an (unknown) open set. If $\Omega = \{u \neq 0\}$ and f is Lipschitz continuous, monotonicity formulas may be used to prove $C^{1,1}$ regularity of u. Nevertheless, this method strongly depends on the Lipschitz continuity of f. Recently, a harmonic analysis technique was developed in [6] to prove optimal regularity under the weakest possible assumption: if the Newtonian potential of f is $C^{1,1}$, then u is uniformly $C^{1,1}$ in $B_{1/2}$, where the bound on the Hessian depends on $\|u\|_{L^\infty(B_1)}$.

Fully nonlinear analogs of (1) have been considered by several researchers. The case

$$F(D^2u) = f\chi_\Omega \quad \text{in } B_1$$

has been studied in [7] for $\Omega = \{u > 0\}$ and in [8] when $\Omega = \{u \neq 0\}$. Moreover, a fully nonlinear version of the method in [6] was developed in [9] and applied to

$$\begin{cases} F(D^2u) = 1 & \text{a.e. in } B_1 \cap \Omega, \\ |D^2u| \leq K & \text{a.e. in } B_1 \backslash \Omega, \end{cases}$$

where Ω is an open set, $K > 0$, and $u \in W^{2,n}(B_1)$. The idea is to replace the projection on second-order harmonic polynomials carried out in [6] with a projection involving the BMO estimates in [10]. For convex operators, this tool is employed to prove that u is $C^{1,1}$ in $B_{1/2}$ and, under a standard thickness assumption, that the free boundary is locally C^1.

Our main result is Theorem 1 and establishes optimal regularity for the more general free boundary problem

$$\begin{cases} F(D^2u, x) = f(x) & \text{a.e. in } B_1 \cap \Omega, \\ |D^2u| \leq K & \text{a.e. in } B_1 \backslash \Omega, \end{cases} \tag{2}$$

where Ω is an open set, $K > 0$, f is Hölder continuous, and under certain structural conditions on F (see §1.1). As a direct consequence, we obtain optimal regularity for general operators $F(D^2u, Du, u, x)$ and thereby address [9, Remark 1.1], see Corollary 2. Free boundary problems of this type appear in the mean field theory of superconducting vortices [3, Introduction] and optimal switching problems [11].

The underlying principle in the proof is to locally apply Caffarelli's elliptic regularity theory [12] to rescaled variants of (2) in order to obtain a bound on D^2u. The main difficulty lies in verifying an average L^n decay of the right-hand side in question. However, one may exploit that $u \in C^{1,\alpha}(B_1)$, D^2u is bounded in $B_1 \backslash \Omega$, and the BMO estimates in [10] to prove that locally around a free boundary point, the coincidence set $B_1 \backslash \Omega$ decays fast enough to ensure the L^n decay. Our assumptions on F involve conditions which enable us to utilize standard tools such as the maximum principle and Evans–Krylov theorem.

Moreover, once we establish that $u \in C^{1,1}$ in $B_{1/2}$, the corresponding regularity theory for the free boundary follows in a standard way through the classification of blow-up solutions and is carried out in §3. Indeed, non-degeneracy holds if f is positive on $\overline{B}_1$ and $\{|\nabla u| \neq 0\} \subset \Omega$. Furthermore, blow-up solutions around thick free boundary points are half-space solutions, and this fact combines with a directional monotonicity result to yield C^1 regularity of the free boundary, see Theorem 15 for a precise statement.

Finally, we generalize the above-mentioned results to the parabolic setting (see also [13]) in §4 by considering the free boundary problem

$$\begin{cases} \mathcal{H}(u(X), X) = f(X) & \text{a.e. in } Q_1 \cap \Omega, \\ |D^2u| \leq K & \text{a.e. in } Q_1 \backslash \Omega, \end{cases}$$

where $X = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, $\mathcal{H}(u(X), X) := F(D^2u(X), X) - \partial_t u(X)$, Q_1 is the parabolic cylinder $B_1(0) \times (-1, 0)$, $\Omega \subset Q_1$ is some open set, and $K > 0$.