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# Behaviour of free boundaries in thin-film flow: The regime of strong slippage and the regime of very weak slippage

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**Abstract**

We analyze the behaviour of free boundaries in thin-film flow in the regime of strong slippage $n \in [1, 2)$ and in the regime of very weak slippage $n \in [\frac{32}{11}, 3)$ qualitatively and quantitatively. In the regime of strong slippage, we construct initial data which are bounded from above by the steady state but for which nevertheless instantaneous forward motion of the free boundary occurs. This shows that the initial behaviour of the free boundary is not determined just by the growth of the initial data at the free boundary. Note that this is a new phenomenon for degenerate parabolic equations which is specific for higher-order equations. Furthermore, this result resolves a controversy in the literature over optimality of sufficient conditions for the occurrence of a waiting time phenomenon. In contrast, in the regime of very weak slippage we derive lower bounds on free boundary propagation which are optimal in the sense that they coincide up to a constant factor with the known upper bounds. In particular, in this regime the growth of the initial data at the free boundary fully determines the initial behaviour of the interface.


*Keywords:* Thin-film equation; Free boundary; Waiting time; Qualitative behaviour; Higher-order parabolic equation; Degenerate parabolic equation

## 1. Introduction

In this paper, we are concerned with the qualitative behaviour of free boundaries in solutions to the thin-film equation

$$\frac{d}{dt}u = -\operatorname{div}(u^n \nabla \Delta u), \quad n \in \mathbb{R}^+,$$

in the case of strong slippage $n \in [1, 2)$ and in the case of very weak slippage $n \in [\frac{32}{11}, 3)$. The thin-film equation describes the evolution of a thin viscous liquid film on a flat solid driven by surface tension. Different values of $n$ correspond to different slip conditions on the fluid–solid interface: The case $n = 3$ corresponds to a no-slip condition (see e.g. [1]), while the case $n = 2$ (or more precisely, $u^n$ replaced by $u^2 + u^3$) corresponds to the Navier slip condition,

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the effective boundary condition for viscous flow on a rough surface [2]. For $n = 1$ the thin-film equation arises as the lubrication approximation of the Hele-Shaw flow [3].

In order to prevent ill-posedness, one needs to prescribe an additional boundary condition at the free boundary. Typically one prescribes the contact angle of solutions. In this paper, we shall be concerned with the case of zero contact angle solutions only; i.e. we formally require $|\nabla u| = 0$ on $\partial \operatorname{supp} u(.,t)$. In the framework of weak solutions with zero contact angle, this condition is enforced by an additional regularity constraint on the solution. For existence of such weak solutions to the thin-film equation with zero contact angle, see the papers [4–8] (note that in the latter works, these solutions are called "strong solutions", as opposed to the weaker solutions of [6] without prescribed contact angle). For a stronger notion of solution (for which existence however is only guaranteed locally or for small initial data and which we shall not be concerned with in the sequel), see the recent works [9–13]. For solution concepts in case of nonzero contact angle and corresponding existence results, see [14–18].

The analysis of qualitative behaviour of solutions to the thin-film equation has a long history. Finite speed of support propagation of solutions has been shown in [5,19–23]. If the initial data are "flat enough" at the free boundary, a waiting time phenomenon occurs [24]: The free boundary of the solution initially does not advance for some time before it starts moving forward. The waiting time has been estimated from below in [25]. However, all now-classical results on qualitative behaviour of weak solutions to the thin-film equation with zero contact angle have been concerned with proving upper bounds on free boundary propagation. Mainly due to the lack of a comparison principle and Harnack inequalities, no lower bounds on free boundary propagation have been available.

With no rigorous lower bounds on free boundary propagation available, there has been a controversy over the optimal condition for the occurrence of a waiting time phenomenon for $n < 2$. In [24], the authors have shown that an estimate of the form $u_0 \lesssim x_+^{\frac{4}{n}}$ is sufficient for a waiting time phenomenon to occur. The authors conjectured their condition to be optimal. In contrast, a formal analysis in [26] suggested the occurrence of a waiting time phenomenon also in case $u_0 \sim x_+^{\beta}$ with $\beta \geq 2$.

Only recently, the author of the present paper has developed a technique for the derivation of lower bounds on interface propagation. For the parameter range $n \in (1, \frac{32}{11})$, the author has shown that for large times the support of solutions spreads at roughly the same rate as the self-similar solution [27]. In the case of weak slippage $n \in (2, \frac{32}{11})$, sufficient conditions for instantaneous propagation of the free boundary in terms of the growth of initial data at the free boundary have been deduced [28]; with a grain of salt, these conditions are the converse of the sufficient conditions for the occurrence of a waiting time phenomenon in [24]. Thus, for $n \in (2, \frac{32}{11})$ the initial behaviour of the free boundary is entirely determined by the growth of the initial data at the free boundary. Nevertheless, the sharp conditions for the nonoccurrence of a waiting time being restricted to $n \geq 2$, the controversy regarding optimality of the sufficient conditions for a waiting time phenomenon in [24] for $n < 2$ has remained unresolved.

These recent results by the author are based on new monotonicity formulas for the thin-film equation of the form

$$\frac{d}{dt} \int u^{1+\alpha} |x - x_0|^{\gamma} \, dx \geq c \int u^{1+\alpha+n} |x - x_0|^{\gamma - 4} \, dx \tag{1}$$

(for certain $\alpha \in (-1, 0)$ and $\gamma < 0$), which hold as long as the support of the solution does not touch the singularity of the weight. Combined with a differential inequality argument due to Chipot and Sideris [29], these formulas imply lower bounds on free boundary propagation. However, the procedure used for obtaining such formulas has been limited to the regime $n \in (1, \frac{32}{11})$. Moreover, in the regime $n \in (1, 2)$ the range of admissible values for $\gamma$ has not been large enough to deduce conditions for instantaneous forward motion of the free boundary which are both necessary and sufficient at the same time.

It is well-known that the qualitative behaviour of solutions to the thin-film equation depends sensitively on the parameter $n$: for $n > 1.5$, no backward motion of the free boundary may happen, while for $n < 1.5$ the support of solutions may shrink. For $n \geq 3$, one expects the support of zero contact angle solutions to be constant in time. This sensitive dependence on the parameter is in contrast to the situation for the second-order analogue of the thin-film equation, the porous medium equation

$$u_t = \nabla \cdot (u^{m-1} \nabla u) \; ;$$

the qualitative behaviour of solutions to the porous medium equation is independent of the parameter $m > 1$.

Thus, it is of interest whether the limitations of the recent results by the author are caused by changes in qualitative behaviour of solutions to the thin-film equation or just by limitations of our technical tools.