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Ann. I. H. Poincaré – AN 33 (2016) 337–395

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Cauchy theory for the gravity water waves system with non-localized initial data

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Received 20 March 2014; received in revised form 13 October 2014; accepted 16 October 2014

Available online 11 November 2014

Abstract

In this article, we develop the local Cauchy theory for the gravity water waves system, for rough initial data which do not decay at infinity. We work in the context of L^2-based uniformly local Sobolev spaces introduced by Kato [22]. We prove a classical well-posedness result (without loss of derivatives). Our result implies also a local well-posedness result in Hölder spaces (with loss of $d/2$ derivatives). As an illustration, we solve a question raised by Boussinesq in [9] on the water waves problem in a canal. We take benefit of an elementary observation to show that the strategy suggested in [9] does indeed apply to this setting.

Keywords: Water-waves; Cauchy problem; Uniformly local Sobolev spaces; Paradifferential calculus

1. Introduction

We are interested in this paper in the free boundary problem describing the motion of an incompressible, irrotational fluid flow moving under the force of gravitation, without surface tension, in case where the initial data are neither localized nor periodic. There are indeed two cases where the mathematical analysis is rather well understood: firstly for periodic initial data (in the classical Sobolev spaces $H^s(\mathbf{T}^d)$) and secondly when they are decaying to zero at infinity (for instance for data in $H^s(\mathbf{R}^d)$ with s large enough). With regards to the analysis of the Cauchy problem, we refer to the recent papers of Lannes [25], Wu [31,32] and Germain, Masmoudi and Shatah [19]. We also refer to the introduction of [2] or [7,10,12,23,26,30,33] for more references. However, one can think to the moving surface of a lake or a canal where the waves are neither periodic nor decaying to zero (see also [16]).

A most natural strategy would be to solve the Cauchy problem in the classical Hölder spaces $W^{k,\infty}(\mathbf{R}^d)$. However even the linearized system at the origin (the fluid at rest) is ill-posed in these spaces (see Remark 2.4 below), and this strategy leads consequently to loss of derivatives. Having this loss of derivatives in mind, the other natural

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1 T.A. was supported by the French Agence Nationale de la Recherche, projects ANR-08-JCJC-0132-01 and ANR-08-JCJC-0124-01.

http://dx.doi.org/10.1016/j.anihpc.2014.10.004
0294-1449/

approach is to work in the framework of L^2 based uniformly local Sobolev spaces, denoted by $H^s_{ul}(\mathbf{R}^d)$. These spaces were introduced by Kato (see [22]) in the analysis of hyperbolic systems. Notice however, that compared to general hyperbolic systems, the water waves system appears to be non-local, which induces new difficulties. This framework appears to be quite natural in our context. Indeed, the uniformly local Sobolev spaces $H^s_{ul}(\mathbf{R}^d)$ contain, in particular, the usual Sobolev spaces $H^s(\mathbf{R}^d)$, the periodic Sobolev spaces $H^s(\mathbf{T}^d)$ (spaces of periodic functions on $\mathbf{R}^d$), the sum $H^s(\mathbf{R}^d) + H^s(\mathbf{T}^d)$ and also the Hölder spaces $W^{s,\infty}(\mathbf{R}^d)$ (and as a by-product of our analysis, we get well-posedness in Hölder spaces, with a loss of derivatives).

The aim of this paper is precisely to prove that the water waves system is locally (in time) well posed in the framework of uniformly local Sobolev spaces. Moreover, following our previous paper [2], the data for which we solve the Cauchy problem are allowed to be quite rough. Indeed we shall assume, for instance, that the initial free surface is the graph of a function which belongs to the space $H^{s+\frac{1}{2}}_{ul}(\mathbf{R}^d)$ for $s > 1 + \frac{d}{2}$. In particular, in terms of Sobolev embedding, the initial free surface is merely $W^{\frac{3}{2},\infty}(\mathbf{R}^d)$ thus may have unbounded curvature. On the other hand this threshold should be compared with the scaling of the problem. Indeed it is known that the water wave system has a scaling invariance for which the critical space for the initial free surface is the space $\dot{H}^{1+\frac{d}{2}}(\mathbf{R}^d)$ (or $W^{1,\infty}(\mathbf{R}^d)$). This shows that we solve here the Cauchy problem for data $\frac{1}{2}$ above the scaling. (Notice that in [3] we prove well-posedness, in the classical Sobolev spaces, $\frac{1}{2} - \frac{1}{12}$ above the scaling when $d \geq 2$ and $\frac{1}{2} - \frac{1}{24}$ when $d = 1$.)

As an illustration of the relevance of this low regularity Cauchy theory in the context of *local* spaces, we solve a question raised by Boussinesq in 1910 [9] on the water waves problem in a canal. In [9], Boussinesq suggested to reduce the water-waves system in a canal to the same system on $\mathbf{R}^3$ with periodic conditions with respect to one variable, by a simple reflection/periodization procedure (with respect to the normal variable to the boundary of the canal). However, this idea remained inapplicable for the simple reason that the even extension of a smooth function on the half line is in general merely Lipschitz continuous (due to the singularity at the origin). As a consequence, even if one starts with a smooth initial domain, the reflected/periodized domain will only be Lipschitz continuous. Here, we are able to take benefit of an elementary (though seemingly previously unnoticed) observation which shows that actually, as soon as we are looking for *reasonably smooth* solutions, the angle between the free surface and the vertical boundary of the canal is a right angle. Consequently, the reflected/periodized domain enjoys additional smoothness (namely up to C^3), which is enough to apply our *rough data* Cauchy theory and to show that the strategy suggested in [9] does indeed apply. This appears to be the first result on Cauchy theory for the water-wave system in a domain with boundary.

The present paper relies on the strategies developed in our previous paper [2] and we follow the same scheme of proof. In Section 7, we develop the machinery of para-differential calculus in the framework of uniformly local spaces that we need later. We think that this section could be useful for further studies in this framework. In Section 3 we prove that the Dirichlet–Neumann operator is well defined in this framework (notice that this fact is not straightforward, see [18,15] for related works), and we give a precise description (including sharp elliptic estimates in very rough domains) on these spaces. In Section 4, we symmetrize the system and prove *a priori* estimates. In Section 5, we prove contraction estimates and well posedness. In Section 6, we give the application to the canal (and swimming pools). Finally, in Appendix A, we prove that in the context of Hölder spaces, the water-waves system linearized on the trivial solution (rest) is *ill posed*.

2. The problem and the result

In this paper we shall denote by $t \in \mathbf{R}$ the time variable and by $x \in \mathbf{R}^d$ (where $d \geq 1$), $y \in \mathbf{R}$, the horizontal and vertical space variables. We work in a fluid domain with free boundary and fixed bottom of the form

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Omega(t)\} \quad \text{where}$$
$$\Omega(t) = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : \eta_*(x) < y < \eta(t, x)\}.$$

Here the free surface is described by η, an unknown of the problem, and the bottom by a given function η_*. We shall only assume that η_* is bounded and continuous. We assume that the bottom is the graph of a function for the sake of simplicity: our analysis applies whenever one has the Poincaré inequality given by Lemma 3.1 below. In the case without bottom, the Dirichlet Neumann operator in the simplest case of a flat interface ($\eta = 0$) is equal to $|D_x|$. It is