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Dynamics of nematic liquid crystal flows: The quasilinear approach

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Abstract

Consider the (simplified) Leslie–Ericksen model for the flow of nematic liquid crystals in a bounded domain $\Omega \subset \mathbb{R}^n$ for $n > 1$. This article develops a complete dynamic theory for these equations, analyzing the system as a quasilinear parabolic evolution equation in an $L_p - L_q$-setting. First, the existence of a unique local strong solution is proved. This solution extends to a global strong solution, provided the initial data are close to an equilibrium or the solution is eventually bounded in the natural norm of the underlying state space. In this case the solution converges exponentially to an equilibrium. Moreover, the solution is shown to be real analytic, jointly in time and space.

Résumé

On considère le modèle de Leslie–Ericksen pour les cristaux liquides nématiques dans un domaine borné $\Omega \subset \mathbb{R}^n$. On obtient une théorie dynamique complète pour ce système, analysé comme une équation d'évolution quasi-linéare dans le cadre $L^p - L^q$. En particulier, on démontre l' existence et l'unicité locales d'une solution forte, qui s'étend en un solution forte globale si les conditions initiales sont près d'un équilibre. De plus, on montre que la solution est analytique réelle en espace et temps.

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Keywords: Nematic liquid crystals; Quasilinear parabolic evolution equations; Regularity; Global solutions; Convergence to equilibria

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1. Introduction

We consider the following system modeling the flow of nematic liquid crystal materials in a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla\pi = -\lambda\,\mathrm{div}\big([\nabla d]^{\mathsf{T}}\nabla d\big) & \text{in } (0,T)\times\Omega,\\ \partial_t d + (u\cdot\nabla)d = \gamma\big(\Delta d + |\nabla d|^2 d\big) & \text{in } (0,T)\times\Omega,\\ \mathrm{div}\, u = 0 & \text{in } (0,T)\times\Omega,\\ (u, \partial_\nu d) = (0,0) & \text{on } (0,T)\times\partial\Omega,\\ (u,d)_{|t=0} = (u_0, d_0) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, the function $u : (0,\infty)\times\Omega \to \mathbb{R}^n$ describes the velocity field, $\pi : (0,\infty)\times\Omega \to \mathbb{R}$ is the pressure, and $d : (0,\infty)\times\Omega \to \mathbb{R}^n$ represents the macroscopic molecular orientation of the liquid crystal. Due to the physical interpretation of d it is natural to impose the condition

$$|d| = 1 \quad \text{in } (0,T)\times\Omega. \tag{1.2}$$

We will show in the following that this condition is indeed preserved by the above system; see Proposition 4.3 below for details.

The constants $\nu > 0$, $\lambda > 0$, and $\gamma > 0$ represent viscosity, the competition between kinetic energy and potential energy and the microscopic elastic relaxation time for the molecular orientation field, respectively. For simplicity, we set $\nu = \lambda = \gamma = 1$, which does not change our analysis.

The continuum theory of liquid crystals was developed by Ericksen and Leslie during the 1950's and 1960's in [10,19]. The Ericksen–Leslie theory is widely used as a model for the flow of liquid crystals, see for example the survey articles by Leslie in [11] and also [4,7,15,22].

The set of Eqs. (1.1) was considered first in [23], however for the situation where in the second equation of (1.1) the term $|\nabla d|^2 d$ is replaced by $f(d) = \nabla F(d)$, i.e.

$$d_t + (u\cdot\nabla)d = \gamma\big(\Delta d - f(d)\big),$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth, bounded function. Note that in this situation, the condition (1.2) *cannot* be preserved in general. Thus, this condition was replaced in [22] and [23] by the Ginzburg–Landau energy functional, i.e. f is assumed to satisfy $f(d) = \nabla F(d) = \nabla\frac{1}{\varepsilon^2}(|d|^2 - 1)^2$. In 1995, Lin and Liu [23] proved the existence of global weak solutions to (1.1) in dimension 2 or 3 under the assumptions that $u_0 \in L_2(\Omega)$, $d_0 \in H^1(\Omega)$, and $d_0 \in H^{3/2}(\partial\Omega)$. Existence and uniqueness of global classical solutions were also obtained by them in dimension 2 provided $u_0 \in H^1(\Omega)$, $d_0 \in H^2(\Omega)$, and provided the viscosity ν is large in dimension 3. For regularity results of weak solutions in the spirit of Caffarelli–Kohn–Nirenberg we refer to [24].

Hu and Wang [16] considered in 2010 the case of $f(d) = 0$ and proved existence and uniqueness of a global strong solution for small initial data in this case. They proved moreover that whenever a strong solutions exist, all global weak solutions as constructed in [23] must be equal to this strong solution. The idea of their approach was to consider the above system (1.1) as a semilinear equation with a forcing term $\lambda\,\mathrm{div}([\nabla d]^{\mathsf{T}}\nabla d)$ on the right-hand side.

The system (1.1) with $f(d) = |\nabla d|^2 d$ was revisited by Lin, Lin, and Wang in 2010. They proved in [21] interior and boundary regularity theorems under smallness condition in dimension 2 and established the existence of global weak solutions on bounded smooth domains $\Omega \subset \mathbb{R}^2$ that are smooth away from a finite set. Furthermore, Wang proved in [33] global well-posedness for this system for initial data being small in $BMO^{-1} \times BMO$ in the case of a whole space, i.e. $\Omega = \mathbb{R}^n$, by combining techniques of Koch and Tataru with methods from harmonic maps to certain Riemannian manifolds.

Let us emphasize at this point that the system (1.1) represents a simplification of the full Ericksen–Leslie system. In particular, stretching and rotational effects of the director field are not taken into account in (1.1). Coutand and Shkoller [6] considered in 2001 a modification of system (1.1) in which the second line of (1.1) is replaced by

$$\partial_t d + u\cdot\nabla d - d\cdot\nabla u = \gamma\left(\Delta d - \frac{1}{\varepsilon^2}\big(|d|^2 - 1\big)d\right) \quad \text{in } (0,T)\times\Omega. \tag{1.3}$$