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# On the inviscid limit of the 2D Navier–Stokes equations with vorticity belonging to BMO-type spaces ☆

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**Abstract**

In a recent paper [6], the global well-posedness of the two-dimensional Euler equation with vorticity in $L^1 \cap \mathrm{LBMO}$ was proved, where LBMO is a Banach space which is strictly imbricated between $L^\infty$ and BMO. In the present paper we prove a global result on the inviscid limit of the Navier–Stokes system with data in this space and other spaces with the same BMO flavor. Some results of local uniform estimates on solutions of the Navier–Stokes equations, independent of the viscosity, are also obtained.


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## 1. Introduction

In this work, we consider the problem of the inviscid limit of the 2D Navier–Stokes equations with rough initial data. More precisely, we are interested in the situation where the vorticity lives in specific Morrey–Campanato spaces (in the same flavor as already studied in [6,4] and very recently in [12]). Morrey–Campanato spaces are Banach spaces which extend the notion of a BMO function (a function with bounded mean oscillation) describing situations where the oscillation of the function in a ball is controlled depending upon the radius of the ball. These spaces have attracted much attention in the last few decades due to their remarkable properties (John–Nirenberg inequalities, duality with Hardy spaces, etc.). For example, the theory of Morrey–Campanato spaces is useful when the Sobolev embedding theorem is not available and has proven to be particularly useful in the study of elliptic PDEs.

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We do not detail the literature about these spaces since it is huge. In this work, we only focus on the $L^\alpha mo$ spaces (see precise definitions in Section 2) where the oscillations of a function on a ball of radius $r \ll 1$ are bounded by $|\log(r)|^{-\alpha}$. What is interesting, is that the scale $(L^\alpha mo)_{0<\alpha<1}$ can be thought of as an intermediate scale between BMO (for $\alpha \to 0$) and $L^\infty$ (for $\alpha \to 1$).

### 1.1. The Navier–Stokes system

The Navier–Stokes system is the basic mathematical model for viscous incompressible flows and reads as follows:

$$(NS_\varepsilon) \quad \begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla P^\varepsilon = 0, \\ \nabla . u^\varepsilon = 0, \\ u^\varepsilon_{|t=0} = u_0. \end{cases} \tag{1.1}$$

Associated to the viscosity parameter $\varepsilon$, the vector field $u^\varepsilon$ stands for the velocity of the fluid, the quantity $P^\varepsilon$ denotes the scalar pressure, and $\nabla . u^\varepsilon = 0$ means that the fluid is incompressible. We also detail the fractional Navier–Stokes equation, of order $\alpha \in (0, 1)$:

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \varepsilon(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon + \nabla P^\varepsilon = 0, \\ \nabla . u^\varepsilon = 0, \\ u^\varepsilon_{|t=0} = u_0, \end{cases} \tag{1.2}$$

where the diffusion term is given by the fractional power of the Laplacian operator. When we neglect the diffusion term, we obtain the Euler equations

$$(E) \quad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = 0, \\ \nabla . u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1.3}$$

The mathematical study of the Navier–Stokes system was initiated by Leray in his pioneering work [24]. In fact, by using a compactness method, he proved that for any divergence-free initial data $v^0$ in the energy space $L^2$, there exists a global solution to $(NS_\varepsilon)$. In *two* dimensions that weak solution was proven to be unique. However, for higher dimensions ($d \geq 3$) the problem of uniqueness is still widely open. In the 60's, Fujita and Kato [19] constructed for initial data lying in the critical Sobolev space $\dot{H}^{\frac{d}{2}-1}$ a class of unique local solutions called mild solutions. We emphasize that the same result holds true when the initial data belongs to the inhomogeneous Sobolev space $H^s$, with $s \geq \frac{d}{2} - 1$. The global existence of these solutions is an outstanding open problem. However a positive answer is given at least in both the following cases: either when the initial data is small in the critical space $\dot{H}^{\frac{d}{2}-1}$ which is invariant under the scaling of the Navier–Stokes equations, or in the space dimension two (this is because in two dimensions the scale invariant space is the energy space).

### 1.2. The Euler system

In the two dimensional space and when the regularity is sufficient to give a sense to the Biot–Savart law, then one can consider an alternative weak formulation: the vorticity-stream weak formulation. It consists in resolving the weak form of (1.3) in terms of vorticity $\omega = \mathrm{curl}(u)$:

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \tag{1.4}$$

supplemented with the Biot–Savart law:

$$u = K * \omega, \quad \text{with } K(x) = \frac{x^\perp}{2\pi |x|^2}.$$

The questions of existence/uniqueness of weak solutions have been extensively studied (see [10,7,25] for instance). We emphasize that, unlike the fixed-point argument, the compactness method does not guarantee the uniqueness of the solutions and then the two issues (existence/uniqueness) are usually dealt with separately. Existence and uniqueness of