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Multi-bubble nodal solutions for slightly subcritical elliptic problems in domains with symmetries

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Abstract

We study the existence of sign-changing solutions with multiple *bubbles* to the slightly subcritical problem

$$-\Delta u = |u|^{2^*-2-\varepsilon}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, $2^* = \frac{2N}{N-2}$ and $\varepsilon > 0$ is a small parameter. In particular we prove that if Ω is convex and satisfies a certain symmetry, then a nodal four-bubble solution exists with two positive and two negative bubbles.

MSC: 35B40; 35J20; 35J65

Keywords: Slightly subcritical problem; Sign-changing solutions; Finite-dimensional reduction; Max–min argument

1. Introduction

We are concerned with the slightly subcritical elliptic problem

$$\begin{cases} -\Delta u = |u|^{2^*-2-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, $\varepsilon > 0$ is a small parameter. Here 2^* denotes the critical exponent in the Sobolev embeddings, i.e. $2^* = \frac{2N}{N-2}$.

In [21] Pohožaev proved that the problem (1.1) does not admit a nontrivial solution if Ω is star-shaped and $\varepsilon \leqslant 0$. On the other hand problem (1.1) has a positive solution if $\varepsilon \leqslant 0$ and Ω is an annulus, see Kazdan and Warner [18].

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In [2] Bahri and Coron found a positive solution to (1.1) with $\varepsilon = 0$ provided that the domain Ω has a *nontrivial topology*. Moreover in [12–14,20] the authors considered the slightly supercritical case where $\varepsilon < 0$ is close to 0 and proved solvability of (1.1) in Coron's situation of a domain with one or more small holes.

In the subcritical case $\varepsilon > 0$ the problem (1.1) is always solvable, since a positive solution u_ε can be found by solving the variational problem

$$\inf\left\{\int_\Omega |\nabla u|^2 \;\middle|\; u \in H_0^1(\Omega),\ \|u\|_{2^*-\varepsilon} = 1\right\}.$$

In [9,16,17,23,24] it was proved that, as $\varepsilon \to 0^+$, the ground state solution u_ε blows up and concentrates at a point ξ which is a critical point of the Robin's function of Ω. In addition to the one-peak solution u_ε, several papers have studied concentration phenomena for positive solutions of (1.1) with multiple blow-up points [3,22]. In a convex domain such a phenomenon cannot occur. Grossi and Takahashi [15] proved the nonexistence of positive solutions for the problem (1.1) blowing up at more than one point. On the other hand, multi-peak nodal solutions always exist for problem (1.1) in a general bounded and smooth domain Ω. Indeed, in [6] a solution with exactly one positive and one negative blow-up point is constructed for the problem (1.1) if $\varepsilon > 0$ is sufficiently small. The location of the two concentration points is also characterized and depends on the geometry of the domain. Moreover the presence of sign-changing solutions with a multiple blow up at a single point has been proved in [19,25] for problem (1.1); such solutions have the shape of towers of alternating-sign bubbles, i.e. they are superpositions of positive bubbles and negative bubbles blowing up at the same point with a different velocity. We also quote the paper [8], where the authors study the blow up of the low energy sign-changing solutions of problem (1.1) and they classify these solutions according to the concentration speeds of the positive and negative part. Finally, we mention the papers [4] and [7] where, by a different approach, the authors provide existence and multiplicity of sign-changing solutions for more general problems than (1.1). These papers are however not concerned with the profile of the solutions.

In this paper we deal with the construction of sign-changing solutions which develop a spike-shape as $\varepsilon \to 0^+$, blowing up positively at some points and negatively at other points, generalizing the double blowing up obtained in [6]. We are able to prove that on certain domains Ω, (1.1) admits solutions with exactly two positive and two negative blow-up points. Moreover, the asymptotic profile of the blow up of these solutions resembles a *bubble*, namely a solution of the equation at the critical exponent in the entire $\mathbb{R}^N$. It is natural to ask about the existence of solutions with k blow-up points, also for $k \neq 2, 4$, and in more general domains. We shall discuss this difficult problem below.

In order to formulate the conditions on the domain Ω, we need to introduce some notation. Let us denote by $G(x, y)$ the Green's function of $-\Delta$ over Ω under Dirichlet boundary conditions; so G satisfies

$$\begin{cases} -\Delta_y G(x, y) = \delta_x(y), & y \in \Omega, \\ G(x, y) = 0, & y \in \partial\Omega, \end{cases}$$

where δ_x is the Dirac mass at x. We denote by $H(x, y)$ its regular part, namely

$$H(x, y) = \frac{1}{(N-2)\sigma_N |x-y|^{N-2}} - G(x, y),$$

where σ_N is the surface measure of the unit sphere in $\mathbb{R}^N$. The value of H on the diagonal, i.e. the function $H(x, x)$, is called the Robin's function of the domain Ω.

Here are our assumptions on Ω.

(A1) $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with a $\mathcal{C}^2$-boundary.
(A2) Ω is invariant under the reflection $(x_1, x') \mapsto (x_1, -x')$ where $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{N-1}$.

For simplicity of notation we write the restrictions of G and H to the x_1-axis as g and h respectively, i.e.

$$g(t, s) = G\big((t, 0, \dots, 0), (s, 0, \dots, 0)\big) \quad \text{and} \quad h(t, s) = H\big((t, 0, \dots, 0), (s, 0, \dots, 0)\big). \tag{1.2}$$

Our last assumption concerning the domain is: