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Pulsating semi-waves in periodic media and spreading speed determined by a free boundary model ☆

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Abstract

We consider a radially symmetric free boundary problem with logistic nonlinear term. The spatial environment is assumed to be asymptotically periodic at infinity in the radial direction. For such a free boundary problem, it is known from [7] that a spreading-vanishing dichotomy holds. However, when spreading occurs, only upper and lower bounds are obtained in [7] for the asymptotic spreading speed. In this paper, we investigate one-dimensional pulsating semi-waves in spatially periodic media. We prove existence, uniqueness of such pulsating semi-waves, and show that the asymptotic spreading speed of the free boundary problem coincides with the speed of the corresponding pulsating semi-wave.

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1. Introduction

We are interested in the evolution of the positive solution $u(t,r)$ $(r=|x|,\ x\in\mathbb{R}^N,\ N\geqslant 1)$, governed by the following diffusive logistic equation with a free boundary:

$$\begin{cases} u_t - d\Delta u = u\big(\alpha(r)-\beta(r)u\big), & t>0,\ 0<r<h(t),\\ u_r(t,0)=0, \qquad u\big(t,h(t)\big)=0, & t>0,\\ h'(t) = -\mu u_r\big(t,h(t)\big), & t>0,\\ h(0)=h_0, \qquad u(0,r)=u_0(r), & 0\leqslant r\leqslant h_0, \end{cases} \tag{1.1}$$

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where $\Delta u = u_{rr} + \frac{N-1}{r} u_r$; $r = h(t)$ is the free boundary to be determined; h_0, μ and d are given positive constants; $u_0 \in C^2([0, h_0])$ is positive in $[0, h_0)$ and $u_0'(0) = u_0(h_0) = 0$; the functions $\alpha(r)$ and $\beta(r)$ are positive and satisfy the following conditions:

$$\begin{cases} \text{(i)} \quad \alpha,\ \beta \in C^{\nu_0}\big([0,\infty)\big) \quad \text{for some } \nu_0 \in (0,1), \\ \text{(ii)} \quad \text{there exist positive } L\text{-periodic functions } a \text{ and } b \text{ in } C^{\nu_0}(\mathbb{R}) \text{ such that} \\ \qquad \lim_{r\to+\infty}\big(\big|\alpha(r)-a(r)\big| + \big|\beta(r)-b(r)\big|\big) = 0. \end{cases} \tag{1.2}$$

Problem (1.1) may be viewed as describing the spreading of a new or invasive species with population density $u(t, |x|)$ over an N-dimensional habitat, which is radially symmetric, heterogeneous and asymptotically space-periodic near infinity in the radial direction. The initial function $u_0(|x|)$ stands for the population in its early stage of introduction. Its spreading front is represented by the free boundary $|x| = h(t)$, which is a sphere $\partial B_{h(t)}$ with radius $h(t)$ growing at a speed proportional to the gradient of the population density at the front: $h'(t) = -\mu u_r(t, h(t))$. (A deduction of this condition based on ecological considerations can be found in [6].) The coefficient functions $\alpha(|x|)$ and $\beta(|x|)$ represent the intrinsic growth rate of the species and its intra-specific competition respectively, and d is the random diffusion rate.

Problem (1.1) was studied recently in [7], and when α, β are positive constants and the space dimension is one, this problem was considered earlier in [10]. In both cases, it was shown that a unique solution pair (u, h) exists, with $u(t, r) > 0$ and $h'(t) > 0$ for $t > 0$ and $0 \leqslant r < h(t)$, and a spreading-vanishing dichotomy holds, namely, a spatial barrier $r = R^*$ exists, such that either

- **Spreading**: the free boundary breaks the barrier at some finite time (i.e., $h(t_0) \geqslant R^*$ for some $t_0 \geqslant 0$), and then the free boundary goes to infinity as $t \to \infty$ (i.e., $\lim_{t\to\infty} h(t) = \infty$), and the population spreads to the entire space and stabilizes at its positive steady-state, or
- **Vanishing**: the free boundary never breaks the barrier ($h(t) < R^*$ for all $t > 0$), and the population vanishes ($\lim_{t\to\infty} u(t, r) = 0$).

Moreover, when spreading occurs, it follows from Theorem 3.6 of [7] that

$$\liminf_{t\to\infty} \frac{h(t)}{t} \geqslant k_*, \qquad \limsup_{t\to\infty} \frac{h(t)}{t} \leqslant k^*$$

for some positive constants k_* and k^* determined by the pairs $(\alpha_\infty, \beta^\infty)$ and $(\alpha^\infty, \beta_\infty)$, respectively, where

$$\alpha_\infty := \liminf_{r\to\infty} \alpha(r), \qquad \beta^\infty := \limsup_{r\to\infty} \beta(r),$$
$$\alpha^\infty := \limsup_{r\to\infty} \alpha(r), \qquad \beta_\infty := \liminf_{r\to\infty} \beta(r).$$

It follows that if both $\lim_{r\to\infty} \alpha(r)$ and $\lim_{r\to\infty} \beta(r)$ exist, then $\lim_{t\to\infty} \frac{h(t)}{t} = k$ exists, and one may regard k as the asymptotic spreading speed.

The main purpose of this paper is to show that under condition (1.2), $\lim_{t\to\infty} \frac{h(t)}{t}$ also exists, and we will use pulsating semi-waves (to be defined below) induced by (1.1) to determine this limit. These semi-waves are solutions of the one-dimensional problem

$$\begin{cases} u_t - du_{xx} = u\big[a(x) - b(x)u\big], & t \in \mathbb{R},\ -\infty < x < h(t), \\ u\big(t, h(t)\big) = 0, \qquad h'(t) = -\mu u_x\big(t, h(t)\big), & t \in \mathbb{R}. \end{cases} \tag{1.3}$$

Asymptotic spreading in spatially periodic environment based on Cauchy problem models has received extensive study recently. The spreading speed in such models is usually determined by the so called pulsating fronts, whose existence, uniqueness and other properties have been investigated by many authors; see [1–5,12,14,16,22] and the references therein for more details. In particular, a pulsating front of the reaction diffusion equation

$$u_t - du_{xx} = u\big[a(x) - b(x)u\big], \quad (t, x) \in \mathbb{R}^2, \tag{1.4}$$

is a solution to this equation of the form $u(t, x) = \Psi(x - ct, x)$, where c (the speed) is a positive constant and the function $\Psi(\xi, x)$ (the profile) is L-periodic in x; moreover, $\lim_{t\to-\infty} u(t, x) = 0$, $\lim_{t\to+\infty} u(t, x)$ is the unique