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Perturbations of quadratic Hamiltonian two-saddle cycles

Lubomir Gavrilov [a], Iliya D. Iliev [b]

[a] *Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS IMT,F-31062 Toulouse Cedex 9, France*
[b] *Institute of Mathematics, Bulgarian Academy of Sciences, Bl. 8, 1113 Sofia, Bulgaria*

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Abstract

We prove that the number of limit cycles which bifurcate from a two-saddle loop of a planar quadratic Hamiltonian system, under an arbitrary quadratic deformation, is less than or equal to three.

1. Introduction

The theory of plane polynomial quadratic differential systems

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y) \end{cases} \tag{1}$$

is one of the most classical branches of the theory of two-dimensional autonomous systems. Despite of the great theoretical interest in studying of such systems, few is known on their qualitative properties. Let $H(2)$ be the maximal number of limit cycles, which such a system can have. It is still not known whether $H(2) < \infty$ (or $H(k) < \infty$ for a polynomial system of degree k). A survey on the state of art until 1966 was given by Coppel [7] where some basic and specific properties of the quadratic systems are discussed.

It was believed for a long time that $H(2) = 3$, see e.g. [29], until Shi Song Ling gave in 1980 his famous example of a quadratic system with four limit cycles [33].

In 1986 Roussarie [30] proposed a local approach to the global conjecture that $H(k) < \infty$, based on the observation that if the cyclicity is infinite, then a limiting periodic set will exist with infinite cyclicity. All possible 121 limiting periodic sets of quadratic systems were later classified in [8].

Of course, it is of interest to compute explicitly the cyclicity of concrete limiting period sets, the simplest one being the equilibrium point. It is another classical result, due to Bautin (1939), which claims that the cyclicity of a singular point of a quadratic system is at most three. The cyclicity of Hamiltonian quadratic homoclinic loops is two [19,21], and for the reversible ones see [18].

In [35], Żołądek claimed that the cyclicity of the Melnikov functions near quadratic triangles (three-saddle loops) or segments (two-saddle loops) is respectively three and two. From this he deduced that the cyclicity of the triangle or the segment itself is also equal to three or two, respectively. As we know now, this conclusion is not always true. Namely, in the perturbed Hamiltonian case, not all limit cycles near a polycycle are "shadowed" by a zero of a Melnikov function. The bifurcation of "alien" limit cycles is a new phenomenon discovered recently by Caubergh, Dumortier

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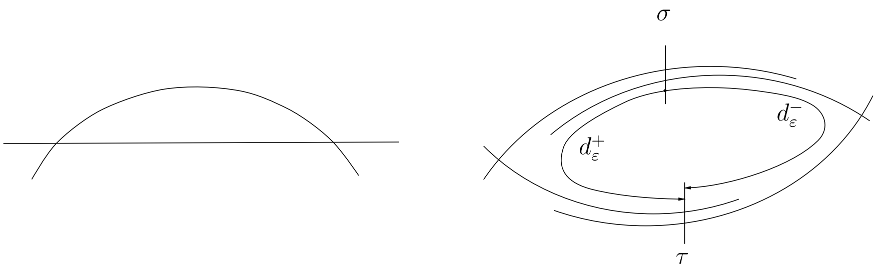

Fig. 1. Monodromic two-saddle loop and the Dulac maps $d_\varepsilon^\pm$.

and Roussarie [3,10]. Li and Roussarie [25] later computed the cyclicity of quadratic Hamiltonian two-loops, when they are perturbed "in a Hamiltonian direction". In the case of a more general perturbation they only noted that "some new approach may be needed".

One of the most interesting developments in this field, starting from the series of papers by Petrovskiĭ and Landis [29], is the proliferation of complex methods, as it can be seen from the 2002 survey of Ilyashenko [23]. A particular interest is given to the study of different infinitesimal versions of the 16th Hilbert problem. Thus, G.S. Petrov [27] used the argument principle to evaluate the zeros of suitable complete Abelian integrals, which on its turn produces an upper bound for the number of limit cycles that a perturbed quadratic system of the form

$$\begin{cases} \dot{x} = y + \varepsilon P(x, y), \\ \dot{y} = x - x^2 + \varepsilon Q(x, y) \end{cases}$$

may have. The result was later generalized for the perturbations of arbitrary generic cubic Hamiltonians in [20,12].

The present paper studies the cyclicity of quadratic Hamiltonian monodromic two-loops, as in Fig. 1. We use complex methods, in the spirit of [14,15], which can also be seen as a far going generalization of the original Petrov method. Our main result is that at most three limit cycles can bifurcate from such a two-loop (Theorem 1), although we did not succeed to prove that this bound is exact. It is interesting to note, that even for a generic quadratic perturbation, two limit cycles can appear near a two-saddle loop, while at the same time the (first) Poincaré–Pontryagin (or Melnikov) function exhibits only one zero. The appearance of the missing alien limit cycle is discussed in Appendix A.

Our semi-local results, combined with the known cyclicity of open period annuli lead also to some global results, formulated in Section 5.

2. Statement of the result

Let X_λ, $\lambda \in \mathbb{R}^{12}$, be the (vector) space of all quadratic planar vector fields, and let X_{λ_0} be a planar quadratic vector field which has two non-degenerate saddle points $S_1(\lambda_0)$, $S_2(\lambda_0)$ connected by two heteroclinic connections Γ_1, Γ_2, which form a monodromic two-loop as in Fig. 1. The union $\Gamma = \Gamma_1 \cup \Gamma_2$ will be referred to as a non-degenerate two-saddle loop. The cyclicity $Cycl(\Gamma, X_\lambda)$ of the two-saddle loop Γ with respect to the deformation X_λ is the maximal number of limit cycles which X_λ can have in an arbitrarily small neighborhood of Γ, as λ tends to λ_0, see [31].

In the present paper we shall suppose in addition, that X_{λ_0} is a Hamiltonian vector field

$$X_{\lambda_0} = X_H\colon \quad \begin{cases} \dot{x} = H_y, \\ \dot{y} = -H_x, \end{cases} \tag{2}$$

where H is a bivariate polynomial of degree three. Our main result is the following

Theorem 1. *The cyclicity of every non-degenerate Hamiltonian two-saddle loop, under an arbitrary quadratic deformation, is at most equal to three.*

The result will be proved by making use of complex methods, as explained in [14], combined with the precise computation of the so-called higher order Poincaré–Pontryagin (or Melnikov) functions, which can be found in [22].