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Smoothing effect of the homogeneous Boltzmann equation with measure valued initial datum

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Abstract

We justify the smoothing effect for measure valued solutions to the space homogeneous Boltzmann equation of Maxwellian type cross sections. This is the first rigorous proof of the smoothing effect for any measure valued initial data except the single Dirac mass, which gives the optimal description on the regularity of solutions for positive time, caused by the singularity in the cross section. The main new ingredient in the proof is the introduction of a time degenerate coercivity estimate by using the microlocal analysis.

Résumé

Nous justifions l'effet régularisant pour les solutions à valeurs mesures de l'équation de Boltzmann spatialement homogène dans le cas des molécules maxwelliennes. Il s'agit de la première preuve rigoureuse de l'effet régularisant pour toutes données initiales à valeurs mesures sauf la masse de Dirac seule, ce qui donne la description optimale de la regularité des solutions en temps positif à causée par la singularité dans le noyau de collision. Le principal ingrédient nouveau dans la preuve est l'introduction d'unc inégalité de coercivité dégénérée par rapport au temps en utilisant l'analyse microlocale.

MSC: primary 35Q20, 76P05; secondary 35H20, 82B40, 82C40

Keywords: Boltzmann equation; Smoothing effect; Measure initial datum; Coercivity estimate

1. Introduction

The purpose of this paper is to analyze the regularizing effect of the Boltzmann equation without angular cutoff in the general setting, that is, for measure valued solutions. Consider the spatially homogeneous Boltzmann equation

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$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{1.1}$$

where $f(t,v)$ is the density distribution of particles with velocity $v \in \mathbb{R}^3$ at time t, and $Q(\cdot,\cdot)$ is the Boltzmann bilinear collision operator given by

$$Q(g,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\big\{g(v'_*)f(v') - g(v_*)f(v)\big\}\,d\sigma\,dv_*,$$

where the conservation of momentum and energy implies that for $\sigma \in \mathbb{S}^2$

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

In the following, we consider the Cauchy problem of (1.1) with a non-negative initial datum

$$f(0,v) = f_0(v). \tag{1.2}$$

Here, $f_0(v)$ is a density of probability distribution (more generally a probability measure).

The non-negative cross section $B(z,\sigma)$ in the collision operator depends only on $|z|$ and the scalar product $\frac{z}{|z|}\cdot\sigma$. Motivated by the physical model of potential of inverse power laws, we assume

$$B\big(|v-v_*|,\cos\theta\big) = \Phi\big(|v-v_*|\big)b(\cos\theta), \quad \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma,\ 0\leqslant\theta\leqslant\frac{\pi}{2},$$

where

$$\Phi\big(|z|\big) = \Phi_\gamma\big(|z|\big) = |z|^\gamma, \quad \text{for some } \gamma > -3, \tag{1.3}$$

$$b(\cos\theta)\theta^{2+2s} \to K \quad \text{when } \theta \to 0+,\ \text{for } 0<s<1 \text{ and } K>0. \tag{1.4}$$

In fact, if the inter-particle potential $U(\rho)$ is proportional to $\rho^{-(q-1)}$ with $q>2$, where ρ denotes the distance between two interacting particles, then s and γ are given by

$$s = 1/(q-1) < 1, \qquad \gamma = 1-4s = 1-4/(q-1) > -3.$$

For this physical model, we have $\gamma = 0$ and $s = 1/4$ when $q = 5$, which is called the Maxwellian molecule. Inspired by this case, in this paper, we consider the Maxwellian molecule type cross section when

$$\gamma = 0, \qquad 0<s<1.$$

The angle θ in the cross section is the deviation angle, i.e., the angle between pre- and post-collisional velocities. Even though the range of θ is in an interval $[0,\pi]$, as in [21], it is customary to restrict it to $[0,\pi/2]$, by replacing $b(\cos\theta)$ by its "symmetrized" version

$$\big[b(\cos\theta) + b\big(\cos(\pi-\theta)\big)\big]\mathbf{1}_{0\leqslant\theta\leqslant\pi/2}$$

because of the invariance of the product $f(v')f(v'_*)$ in the collision operator $Q(f,f)$ under the change of variables $\sigma \to -\sigma$.

One of the important feature of the cross section without angular cutoff is that $b(\cos\theta)$ has the integrable singularity, that is,

$$\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)d\sigma = 2\pi\int_0^{\pi/2} b(\cos\theta)\sin\theta\,d\theta = \infty.$$

This kind of singularity leads to some difficulties in the study of the existence and solution behavior because the gain and loss terms in the collision operator cannot be considered separately. Moreover, the angular singularity also leads to the gain of regularity in the solution. For later analysis, as before, the case where $0<s<1/2$, that is, $\int_0^{\pi/2}\theta b(\cos\theta)\sin\theta\,d\theta < \infty$ is called the mild singularity, and another case $1/2 \leqslant s < 1$ is called the strong singularity. Note that to handle the strong singularity, some symmetry property of the collision operator should be used.