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Gradient integrability and rigidity results for two-phase conductivities in two dimensions

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Abstract

This paper deals with higher gradient integrability for σ-harmonic functions u with discontinuous coefficients σ, i.e. weak solutions of $\operatorname{div}(\sigma \nabla u) = 0$ in dimension two. When σ is assumed to be symmetric, then the optimal integrability exponent of the gradient field is known thanks to the work of Astala and Leonetti and Nesi. When only the ellipticity is fixed and σ is otherwise unconstrained, the optimal exponent is established, in the strongest possible way of the existence of so-called exact solutions, via the exhibition of optimal microgeometries.

We focus also on two-phase conductivities, i.e., conductivities assuming only two matrix values, σ_1 and σ_2, and study the higher integrability of the corresponding gradient field $|\nabla u|$ for this special but very significant class. The gradient field and its integrability clearly depend on the geometry, i.e., on the phases arrangement described by the sets $E_i = \sigma^{-1}(\sigma_i)$. We find the optimal integrability exponent of the gradient field corresponding to any pair $\{\sigma_1, \sigma_2\}$ of elliptic matrices, i.e., the worst among all possible microgeometries.

We also treat the unconstrained case when an arbitrary but finite number of phases are present.

MSC: 30C62; 35B27

Keywords: Beltrami system; Quasiconformal mappings; Elliptic equations; Composites; Gradient integrability

Contents

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1. Introduction

Let Ω be a bounded, open and simply connected subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. We are interested in elliptic equations in divergence form with L^∞ coefficients, specifically,

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega. \tag{1.1}$$

Here σ is a matrix valued coefficient, referred to as *conductivity*, and any weak solution $u \in H^1_{\mathrm{loc}}(\Omega)$ to the equation is called a σ*-harmonic* function. The case of discontinuous conductivities σ is particularly relevant in the context of non-homogeneous and composite materials. With this motivation, we only assume ellipticity. Denote by $\mathbb{M}^{2\times2}$ the space of real 2×2 matrices and by $\mathbb{M}^{2\times2}_{sym}$ the subspace of symmetric matrices.

Definition 1.1. Let $\lambda \in (0, 1]$. We say that $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times2})$ belongs to the class $\mathcal{M}(\lambda, \Omega)$ if it satisfies the following uniform bounds

$$\sigma\xi \cdot \xi \geqslant \lambda|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega, \tag{1.2}$$

$$\sigma^{-1}\xi \cdot \xi \geqslant \lambda|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega. \tag{1.3}$$

We denote by $\mathcal{M}_{sym}(\lambda, \Omega)$ the set of functions in $\mathcal{M}(\lambda, \Omega)$ which are a.e. symmetric.

Finally, we say that σ is elliptic if it belongs to the class $\mathcal{M}(\lambda, \Omega)$ for some positive λ.

The reader may wonder why to use the notion of ellipticity given in Definition 1.1. The reason is the interest in one class of applications related to the theory of the so-called composite materials. Physically this takes into account the possible presence of several well separated length scales. From the mathematical point of view one is forced to consider sequences of problems of type (1.1) and to study the limiting equation in a sense that has later been called homogenization. This process has been first undertaken, historically, in the case of symmetric conductivities giving rise to the notion of G-limit. Later, the study has been extended to the non-necessarily symmetric case and called H-convergence. It is exactly at this point that Murat and Tartar (see [18]) observed that only the ellipticity given in Definition 1.1 has the property to give H-stability. In other words, a class of pdes with uniform bounds of the latter type, H-converge to a pde with the same ellipticity as opposed to what happens if different notions of ellipticity are assumed. For a detailed explanation related to the relationship between composite materials and H-convergence we refer the reader to [2].

It is well known that the gradient of σ-harmonic functions locally belongs to some L^p with $p > 2$. Any σ-harmonic function u can be seen as the real part of a complex map $f : \Omega \mapsto \mathbb{C}$ which is a H^1_{loc} solution to the *Beltrami equation*

$$f_{\bar z} = \mu f_z + \nu \overline{f_z}, \quad \text{in } \Omega, \tag{1.4}$$

where the so-called complex dilatations μ and ν, both belonging to $L^\infty(\Omega, \mathbb{C})$, are given by

$$\mu = \frac{\sigma_{22} - \sigma_{11} - i(\sigma_{12} + \sigma_{21})}{1 + \operatorname{tr}\sigma + \det\sigma}, \qquad \nu = \frac{1 - \det\sigma + i(\sigma_{12} - \sigma_{21})}{1 + \operatorname{tr}\sigma + \det\sigma}, \tag{1.5}$$