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On Clark's theorem and its applications to partially sublinear problems

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Abstract

In critical point theory, Clark's theorem asserts the existence of a sequence of negative critical values tending to 0 for even coercive functionals. We improve Clark's theorem, showing that such a functional has a sequence of critical points tending to 0. Our result also gives more detailed structure of the set of critical points near the origin. An extension of Clark's theorem is also given. Our abstract results are powerful in applications, and thus lead to much stronger results than those in the literature on existence of infinitely many solutions for partially sublinear problems such as elliptic equations and Hamiltonian systems.

Résumé

En théorie des points critiques, le théorème de Clark assure l'existence d'une suite de valeurs critiques négatives tendant vers 0 pour des fonctionnelles paires et coercitives. Nous étendons le théorème de Clark en montrant qu'une telle fonctionnelle possède une suite de points critiques tendant vers 0. Notre résultat permet aussi une description plus précise de l'ensemble des points critiques autour de l'origine. Une extension du théorème de Clark est aussi donnée. Nos résultats abstraits s'avèrent puissants dans les applications et conduisent à des résultats nouveaux concernant l'existence d'une infinité de solutions pour des problèmes partiellement sous linéaires comme des équations elliptiques ou des systèmes hamiltoniens.

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1. Introduction

The Clark theorem [6] is an important tool in critical point theory and is constantly and effectively applied to sublinear differential equations with symmetry. The purpose of this paper is to investigate the structure of the set of critical points given in critical point theorems of the Clark type and to prove existence of infinitely many solutions to partially sublinear problems including partially sublinear elliptic equations and partially sublinear Hamiltonian systems. A variant of the Clark Theorem was given by Heinz in [11].

Theorem A. *Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$. Assume Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$, then Φ has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$.*

Theorem A asserts the existence of a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$, without giving any information on the structure of the set of critical points. In applications to nonlinear boundary value problems, if one knows in addition that there exist u_k with $\Phi(u_k) = c_k$, $\Phi'(u_k) = 0$ such that $\|u_k\| \to 0$ as $k \to \infty$ then the behavior of the nonlinearity for $|u|$ large may not be needed to assert the existence of infinitely many critical points. This idea was first explored in [19] for a variety of nonlinear boundary value problems. For example, consider the classical Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and $f(x,u)$ and $F(x,u) = \int_0^u f(x,t)dt$ satisfy

(a1) $f \in C(\bar{\Omega} \times (-\delta, \delta), \mathbb{R})$ for some $\delta > 0$, $f(x,u)$ is odd in u, and uniformly in $x \in \bar{\Omega}$

$$\lim_{|u| \to 0} \frac{F(x,u)}{u^2} = +\infty,$$

(a2) $2F(x,u) - f(x,u)u > 0$ for $x \in \bar{\Omega}$ and $u \in (-\delta, \delta)$, $u \neq 0$.

Then it was shown in [19] that the problem has a sequence of solutions $u_k \neq 0$ with $\|u_k\| \to 0$ as $k \to \infty$. In applying Theorem A, the nonlinear function f is extended to $\bar{\Omega} \times \mathbb{R}$ so that the resulting variational functional satisfies the conditions of Theorem A. In fact only (a1) is needed in verifying the assumptions of Theorem A. Then a sequence of critical values $c_k < 0$, $c_k \to 0$ as $k \to \infty$, is obtained. (a2) is used to guarantee that the associated functional Φ has $u = 0$ as the only critical point with the critical value 0. Therefore, if (u_k) are critical points corresponding to c_k then (u_k) is a (PS) sequence and it has a convergent subsequence which must converge to 0 with respect to the Sobolev norm $\|\cdot\|$. Finally a regularity argument shows $\|u_k\|_{L^\infty} \to 0$ as $k \to \infty$ so for k large, (u_k) are solutions of the original equation. This unified and generalized several results from [10,1,3] under various additional technical conditions (see Section 3.1 for details).

A very interesting question arising from applications of Theorem A is whether there is a sequence of critical points u_k such that $\Phi(u_k) \to 0$ and $\|u_k\| \to 0$ as $k \to \infty$ under the assumptions of Theorem A. If this question has a positive answer then, when for example applying the abstract result to (1.1), (a2) is completely unnecessary and (a1) can be much weakened. We shall give this question a positive answer and our result gives the structure of the set of critical points near the original in the abstract setting of Clark's theorem. Our first abstract result is the following.

Theorem 1.1. *Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$. Assume Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$, then at least one of the following conclusions holds.*

(i) *There exists a sequence of critical points $\{u_k\}$ satisfying $\Phi(u_k) < 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$.*
(ii) *There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that $\|u\| = a$ and $\Phi(u) = 0$.*