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Entropy conditions for scalar conservation laws with discontinuous flux revisited

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Abstract

We propose new entropy admissibility conditions for multidimensional hyperbolic scalar conservation laws with discontinuous flux which generalize one-dimensional Karlsen–Risebro–Towers entropy conditions. These new conditions are designed, in particular, in order to characterize the limit of vanishing viscosity approximations. On the one hand, they comply quite naturally with a certain class of physical and numerical modeling assumptions; on the other hand, their mathematical assessment turns out to be intricate.

The generalization we propose is not only with respect to the space dimension, but mainly in the sense that the "crossing condition" of Karlsen, Risebro, and Towers (2003) [31] is not mandatory for proving uniqueness with the new definition. We prove uniqueness of solutions and give tools to justify their existence via the vanishing viscosity method, for the multi-dimensional spatially inhomogeneous case with a finite number of Lipschitz regular hypersurfaces of discontinuity for the flux function.

MSC: 35L65; 35L67

Keywords: Inhomogeneous scalar conservation law; Discontinuous flux; Entropy solution; Vanishing viscosity approximation; Well-posedness; Crossing condition

1. Introduction

Conservation laws of the form

$$\partial_t u + \operatorname{div}_{\mathbf{x}} \mathfrak{f}(t, \mathbf{x}, u) = S(t, \mathbf{x}, u) \tag{1}$$

serve as mathematical models for one-dimensional gas dynamics, road traffic, for flows in porous media with neglected capillarity effects, blood flow, radar shape-from-shading problems, and in several other applications. The

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multi-dimensional conservation law also appears in coupled models, although in this case the regularity of the flux function $\mathfrak{f}$ in $(t, \mathbf{x})$ is often not sufficient to develop a full well-posedness theory. The mathematical theory of (1) is very delicate because, in general, even for regular data classical solutions need not to exist; on the other hand, weak (distributional) solutions are, in general, not unique. The classical theory is best established for Cauchy and boundary-value problems in the case where $\mathfrak{f}$ is Lipschitz continuous in $(t, \mathbf{x})$ and uniformly locally Lipschitz continuous in u. The source S can be assumed, e.g., Lipschitz continuous in u uniformly in $(t, \mathbf{x})$. In this case the S.N. Kruzhkov definition of entropy solution in [33] and the associated analysis techniques (vanishing viscosity approximation for the existence proof, doubling of variables for the uniqueness proof) provide a well-posedness framework for (1).

1.1. Discontinuous-flux models and rough entropy inequalities

Local Lipschitz assumption on $k \mapsto \mathfrak{f}(t, \mathbf{x}, k)$ is natural in many applications, but the assumption of regular dependence on the spatial variable $\mathbf{x}$ is very restrictive. Indeed, road traffic with variable number of lanes [19], Buckley–Leverett equation in a layered porous medium (see [29,8]), sedimentation applications (see [23,24,17]) make appear models with piecewise regular, jump discontinuous in $\mathbf{x}$ flux functions. The theory of such problems, called *discontinuous-flux conservation laws*, has been an intense subject of research in the last twenty years. The main goal of this research was to design a suitable approach to definition and numerical approximation of entropy solutions, in relation with the physical context of different discontinuous-flux models. Speaking of "notion of solution" in this context, one usually means weak solutions subject to some additional *admissibility conditions*, cf. [33].

Almost all admissibility conditions designed in the literature were confined to the one-dimensional case. We mention here the minimal jump condition [26], minimal variation condition and Γ-condition [23,24], entropy conditions [31,2], vanishing capillarity limit [29,8], admissibility conditions via adapted entropies [12,18] or via admissible jumps description at the interface [3,4,25]. An extensive overview on the subject as well as a kind of unification of the mentioned approaches in the one-dimensional case was given in [11], where further references can be found.

Adimurthi et al. [3] observed that infinitely many different, though equally mathematically consistent, notions of solution may co-exist in the discontinuous-flux problems; therefore the choice of solution notion is a part of modeling procedure (see, e.g., [8] for an exhaustive study of the vanishing capillarity limits of the one-dimensional Buckley–Leverett equation, where different sets of admissibility conditions are put forward for different choices of physically relevant vanishing capillarity). In the present contribution we limit our attention to characterization of vanishing viscosity limit solutions for problems of kind (1); these approximations were studied in a huge number of works (see, e.g., [26,23,24,42,43,41,31,32,13,18,40,25]) including several works in multiple space dimensions (see [28,27,10,39,16]) and they remain relevant in several models based on discontinuous-flux conservation laws.

The basis of the different definitions of admissibility of solutions is provided by Kruzhkov entropy inequalities [33] in the regions of smoothness of the flux; the main difficulty consists in taking into account the jump discontinuities of the flux. To do so, for a quite general setting one may only assume that

$$\text{for all } k \in \mathbb{R}, \quad \mathfrak{f}(.,.,k) \in BV_{loc}\big(\mathbb{R}^+ \times \mathbb{R}^d\big). \tag{2}$$

This rather weak regularity appears naturally e.g. in the study of triangular systems of conservation laws (see [30] and references therein). In the framework (2), under a non-degeneracy assumption of the fluxes $k \mapsto \mathfrak{f}(t, \mathbf{x}, k)$, existence of solutions satisfying the family of entropy inequalities

$$\begin{aligned} \forall k \in \mathbb{R} \quad & |u-k|_t + \operatorname{div}_{\mathbf{x}}\big(\operatorname{sgn}(u-k)\big(\mathfrak{f}(t,\mathbf{x},u) - \mathfrak{f}(t,\mathbf{x},k)\big)\big) - \operatorname{sgn}(u-k) S(t,\mathbf{x},u) \\ & \le -\operatorname{sgn}(u-k)\big(\operatorname{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},k)\big)^{ac} + \big|\big(\operatorname{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},k)\big)^{s}\big| \quad \text{in } \mathcal{D}'\big(\mathbb{R}^+ \times \mathbb{R}^d\big) \end{aligned} \tag{3}$$

has been proved by Panov in [40] using vanishing viscosity method; here,

$$\operatorname{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},k) = \big(\operatorname{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},k)\big)^{ac} + \big(\operatorname{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},k)\big)^{s}$$

is the Jordan decomposition of the Radon measure $\operatorname{div}_{\mathbf{x}} \mathfrak{f}(t, \mathbf{x}, k)$ into its absolutely continuous part and its singular part (cf. (17) below, for a particular but representative case). Let us stress that inequalities (3) use a roughly estimated contribution of the jump singularities in the flux $\mathfrak{f}$, which turns out to be a serious obstacle for proving uniqueness of solutions in the sense (3).