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Existence and uniqueness of optimal transport maps

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Abstract

Let (X, d, m) be a proper, non-branching, metric measure space. We show existence and uniqueness of optimal transport maps for cost written as non-decreasing and strictly convex functions of the distance, provided (X, d, m) satisfies a new weak property concerning the behavior of m under the shrinking of sets to points, see Assumption 1. This in particular covers spaces satisfying the measure contraction property.

We also prove a stability property for Assumption 1: If (X, d, m) satisfies Assumption 1 and $\tilde{m} = g \cdot m$, for some continuous function $g > 0$, then also $(X, d, \tilde{m})$ verifies Assumption 1. Since these changes in the reference measures do not preserve any Ricci type curvature bounds, this shows that our condition is strictly weaker than measure contraction property.

Keywords: Optimal transport; Existence of maps; Uniqueness of maps; Measure contraction property

1. Introduction

In [10], Gaspard Monge studied the by now famous minimization problem

$$\inf_{T_\sharp \mu_0 = \mu_1} \int d\big(x, T(x)\big)\mu_0(dx), \tag{1.1}$$

on Euclidean space, where μ_0 and μ_1 are two given probability measures and the minimum is taken over all maps pushing μ_0 forward to μ_1. This problem turned out to be very difficult because the functional is non-linear and the constraint set maybe empty. 70 years ago, Kantorovich [8] came up with a relaxation of the minimization problem (1.1). He allowed arbitrary couplings q of the two measures μ_0 and μ_1, which we denote by the set $\Pi(\mu_0, \mu_1)$, and also more general cost functions $c : X \times X \to \mathbb{R}$:

$$\inf_{q \in \Pi(\mu_0,\mu_1)} \int c(x, y)q(dx, dy). \tag{1.2}$$

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Minimizers of (1.2) are called optimal couplings and, therefore, this family of problems is commonly called optimal transport problems. A natural and interesting question is under which conditions do these two minimization problems coincide, i.e. under which conditions is the optimal coupling given by a transportation map. In [5], Brenier showed using ideas from fluid dynamics that on Euclidean space with cost function $c(x, y) = |x - y|^2$ there is always a unique optimal transportation map as soon as μ_0 is absolutely continuous with respect to the Lebesgue measure. Soon after, McCann [9] generalized this result to Riemannian manifolds with more general cost functions including convex functions of the distance. By now, this result is shown in a wide class of settings, for instance for non-decreasing strictly convex functions of the distance in Alexandrov spaces [3], for squared distance on the Heisenberg group [2], and recently for the squared distance on $\mathsf{CD}(K, N)$ and $\mathsf{CD}(K, \infty)$ spaces by Gigli [7] and for squared distance cost by Ambrosio and Rajala in a metric Riemannian like framework [1].

In this paper we show existence and uniqueness of optimal transport maps on proper, non-branching, metric measure spaces satisfying a new condition, Assumption 1, for cost functions of the form $c(x, y) = h(d(x, y))$, with h strictly convex and non-decreasing.

Assumption 1 does not imply any lower curvature bounds in the sense of Lott, Sturm and Villani. In particular in Section 3 we prove that Assumption 1 cannot imply the measure contraction property, MCP. On the other hand the measure contraction property implies Assumption 1. Therefore our result applies to spaces enjoying MCP, recovers most of the previously mentioned results and in many cases also extends them.

To our knowledge this is the first existence result of optimal maps in metric spaces for $c(x, y) = h(d(x, y))$, with h strictly convex and non-decreasing with no assumption on a lower bound on the Ricci curvature of the space. For $h = id$, existence of optimal maps, again with no assumption on the curvature of the metric space, has been obtained in [4].

The crucial idea for the proof of the main result is to *approximate the c-cyclically monotone set* on which the optimal measure is concentrated by means of a suitably chosen sequence of c-cyclically monotone sets representing transports into a discrete target.

We conclude this Introduction by describing the structure of the paper. In Section 2 we introduce the general setting of the paper, define Assumption 1 and state the two main results: the existence of optimal transport maps (Theorem 2.1) and the stability under changes in the reference measure of (X, d, m) of Assumption 1 (Theorem 2.2). In Section 3 we prove Theorem 2.2 while Section 4 and Section 5 are devoted to the proof of Theorem 2.1.

2. Notation and main result

We now introduce the setting of this article. If not explicitly stated otherwise we will always assume to work in this framework.

Let (X, d, m) be a proper, non-branching, metric measure space, that is

- (X, d) is a proper, complete and separable metric space with a non-branching geodesic structure;
- m is a positive Borel measure, finite over compact sets whose support coincides with X.

In case we drop the proper assumption, we will refer to (X, d, m) just as non-branching metric measure space. Let μ_0, μ_1 be probability measures over X and let $h : [0, \infty) \to [0, \infty)$ be a strictly convex and non-decreasing map.

We study the following minimization problem

$$\min_{T_\sharp \mu_0 = \mu_1} \int h\big(d\big(x, T(x)\big)\big)\mu_0(dx), \tag{2.1}$$

where $T_\sharp \mu_0$ denotes the push forward of μ_0 under the map T. In the sequel, we will often denote the cost function $h \circ d$ just with c. To get hands on the minimization problem (2.1) we also study its relaxed form, the Kantorovich problem. Let $\Pi(\mu_0, \mu_1)$ be the set of transference plans, i.e.

$$\Pi(\mu_0, \mu_1) := \big\{\pi \in \mathcal{P}(X \times X) : (P_1)_\sharp \pi = \mu_0, (P_2)_\sharp \pi = \mu_1\big\},$$

where $P_i : X \times X \to X$ is the projection map onto the i-th component, $P_i(x_1, x_2) = x_i$ for $i = 1, 2$.