Available online at www.sciencedirect.com
SciVerse ScienceDirect
Ann. I. H. Poincaré – AN 31 (2014) 349–368
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Flowing maps to minimal surfaces: Existence and uniqueness of solutions

Melanie Rupflin *

Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, 14476 Potsdam, Germany

Received 10 July 2012; accepted 31 March 2013

Available online 16 April 2013

Abstract

We study the new geometric flow that was introduced in the paper [12] of Topping and the author that evolves a pair of map and (domain) metric in such a way that it changes appropriate initial data into branched minimal immersions. In the present paper we focus on the existence theory as well as the issue of uniqueness of solutions. We establish that a (weak) solution exists for as long as the metrics remain in a bounded region of moduli space, i.e. as long as the flow does not collapse a closed geodesic in the domain manifold to a point. Furthermore, we prove that this solution is unique in the class of all weak solutions with non-increasing energy. This work complements the paper of Topping and the author [12] where the flow was introduced and its asymptotic convergence to branched minimal immersions is discussed.

Résumé

Nous étudions le nouveau flot géométrique introduit dans l'article [12] de Topping et de l'auteure, qui transforme un couple formé d'une application d'une surface vers une variété riemannienne et d'une métrique riemannienne du domaine. Ce flot change des données initiales appropriées en des immersions minimales ramifiées. Nous prouvons qu'une solution faible existe tant que le flot ne contracte pas une géodésique fermée du domaine en un point. De plus, nous montrons que cette solution est unique dans la classe des solutions faibles avec énergie décroissante. Ce travail complète l'article de Topping et de l'auteure [12] où le flot est introduit et où sa convergence asymptotique est étudiée.

1. Introduction

Let M be a smooth closed orientable surface and let (N, G_N) be a (fixed) closed smooth Riemannian manifold of arbitrary dimension that we view as being isometrically immersed in $\mathbb{R}^K$ for some $K \in \mathbb{N}$.

For g a Riemannian metric on M and a map $u : (M, g) \to (N, G_N)$ the Dirichlet energy is defined as

$$E(u, g) := \frac{1}{2} \int_M |du|^2 \, dv_g.$$

* Current address: Institut für Differentialgeometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany. Tel.: +49 511 762 2305.

E-mail address: melanie.rupflin@math.uni-hannover.de.

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.03.008

We remark that (u, g) is a critical point of E if and only if u is harmonic and weakly conformal, i.e. a branched minimal immersion or a constant map. In the present paper we establish the existence theory for the natural gradient flow of E (considered as a function of both the map and the domain metric) which was introduced in [12]. We refer to this joint paper of Topping and the author for the construction and the geometric background of this flow, but for convenience here recall the main points that led to the definition in [12].

We consider the negative gradient flow of E considered as a function of both the map and the domain metric, but taking into account the symmetries of E, that is the invariance under conformal variations of the domain as well as under the pull-back by diffeomorphisms applied simultaneously to the metric and the map component. That is we consider E and its gradient flow on the set

$$\mathcal{A} = \{[(u, g)];\ g \in \mathcal{M}_c,\ u \in C^\infty(M, N)\}$$

of equivalence classes where we identify $(u, g) \sim (u \circ f, f^*g)$ for smooth diffeomorphisms $f : M \to M$ homotopic to the identity. Here $\mathcal{M}_c$ stands for the set of smooth metrics of constant (Gauss-)curvature $c = 1, 0, -1$ for surfaces of genus $\gamma = 0, 1$ respectively $\gamma \geqslant 2$, with unit area in case $\gamma = 1$.

The tangent space of $\mathcal{M}_c$ splits orthogonally into a horizontal part consisting of the real parts of holomorphic quadratic differentials and a vertical part along the fibres of the action of diffeomorphisms on $\mathcal{M}_c$, i.e. the space of Lie-derivatives of the metric, compare Lemma 2.5 below. This canonical splitting allows us in [12] to represent solutions of the L^2-negative gradient flow of E on $\mathcal{A}$ by the solutions of the system

$$\partial_t u = \tau_g(u), \tag{1.1a}$$

$$\frac{dg}{dt} = \frac{\eta^2}{4} \operatorname{Re}\big(P_g^H(\Phi(u, g))\big). \tag{1.1b}$$

Here $\tau_g(u) = \operatorname{tr}_g(\nabla_g(du)) = \Delta_g u + A_g(u)(\nabla u, \nabla u)$, A the second fundamental form of $N \hookrightarrow \mathbb{R}^K$, denotes the tension field of $u : (M, g) \to (N, G_N)$ and $\Phi(u, g)$ stands for the Hopf-differential, i.e. the quadratic differential given in conformal coordinates $z = x + iy$ of (M, g) as $\Phi(u, g) = \phi\, dz^2$ for $\phi = |u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle$. Furthermore P_g^H denotes the L^2-orthogonal projection from the space of quadratic differentials onto the finite dimensional subspace of *holomorphic* quadratic differentials on (M, g). Finally $\eta > 0$ is a free coupling constant related to the choice of L^2-metric on $\mathcal{A}$.

As the main result of this paper we prove the following existence and uniqueness theorem.

Theorem 1.1. *To any given initial data $(u_0, g_0) \in C^\infty(M, N) \times \mathcal{M}_c$ there exists a weak solution (u, g) of (1.1) defined on a maximal interval $[0, T)$, $T \leqslant \infty$, that satisfies the following properties*:

- (i) *The solution (u, g) is smooth away from at most finitely many singular times $T_i \in (0, T)$ at which 'harmonic spheres bubble off'. More precisely as $t \nearrow T_i$ energy concentrates at a finite number of points $S(T_i) \subset M$ and suitable rescalings of the maps $u(t)$ around points in $S(T_i)$ converge as $t \nearrow T_i$ to (a bubble-tree of) non-trivial harmonic maps from $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ to N.*
- (ii) *As $t \to T_i$ the maps $u(t)$ converge weakly in H^1 and smoothly away from the set $S(T_i)$ to a limit $u(T_i) \in H^1(M, N)$. Furthermore, the metrics $g(t)$ converge smoothly to an element $g(T) \in \mathcal{M}_c$; in fact, the flow of metrics is Lipschitz-continuous with respect to all C^m metrics on $\mathcal{M}_c$ across singular times.*
- (iii) *The energy $t \mapsto E(u(t), g(t))$ is non-increasing.*
- (iv) *The solution exists as long as the metrics do not degenerate in moduli space; i.e. either $T = \infty$ or the length $\ell(g(t))$ of the shortest closed geodesic in $(M, g(t))$ converges to zero as $t \nearrow T$.*

Furthermore, the solution is uniquely determined by its initial data in the class of all weak solutions with non-increasing energy.

Definition 1.2. We call $(u, g) \in H^1_{loc}(M \times [0, T), N) \times C^0([0, T), \mathcal{M}_{-1})$ a weak solution of (1.1) if u solves (1.1a) in the sense of distributions and if g is piecewise C^1 (viewed as a map from $[0, T)$ into the space of symmetric $(0, 2)$ tensors equipped with any C^k metric, $k \in \mathbb{N}$) and satisfies (1.1b) away from times where it is not differentiable.