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Multi-bang control of elliptic systems

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Abstract

Multi-bang control refers to optimal control problems for partial differential equations where a distributed control should only take on values from a discrete set of allowed states. This property can be promoted by a combination of L^2 and L^0-type control costs. Although the resulting functional is nonconvex and lacks weak lower-semicontinuity, application of Fenchel duality yields a formal primal-dual optimality system that admits a unique solution. This solution is in general only suboptimal, but the optimality gap can be characterized and shown to be zero under appropriate conditions. Furthermore, in certain situations it is possible to derive a generalized multi-bang principle, i.e., to prove that the control almost everywhere takes on allowed values except on sets where the corresponding state reaches the target. A regularized semismooth Newton method allows the numerical computation of (sub)optimal controls. Numerical examples illustrate the effectiveness of the proposed approach as well as the structural properties of multi-bang controls.

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1. Introduction

This work is concerned with the problem

$$\begin{cases} \min\limits_{u,y} \dfrac{1}{2}\|y-z\|_{L^2}^2 + \dfrac{\alpha}{2}\|u\|_{L^2}^2 + \beta \displaystyle\int_\Omega \prod_{i=1}^d |u(x)-u_i|_0 \, dx \\ \text{s.t.} \quad Ay = u, \quad u_1 \leqslant u(x) \leqslant u_d \text{ for almost every } x \in \Omega \end{cases} \tag{1.1}$$

for given $\alpha > 0$, $\beta > 0$, real numbers $u_1 < \cdots < u_d$, $d \geqslant 2$, and a target $z \in L^2(\Omega)$. We assume that $A : V \to V^*$ is an isomorphism for a Hilbert space V with continuous, compact and dense embeddings $V \hookrightarrow L^2(\Omega) \hookrightarrow V^*$ (typically, an elliptic partial differential operator). The *binary* term

$$|t|_0 := \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t \neq 0, \end{cases}$$

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is related to Donoho's counting measure. Problem (1.1) is motivated by optimal control problems where it is only possible or desired for the control to take on values from a discrete set of given *control states* u_i (e.g., velocities or voltages), preferably those of smallest possible magnitude. In analogy to bang-bang controls, which (under suitable conditions) attain their control constraints almost everywhere, we refer to such controls as *multi-bang controls*.

Let us remark on some related control problems. For $d = 1$, $u_1 = 0$, and no control constraints, problem (1.1) was investigated in [1], where the choice of the cost was motivated by obtaining sparsity in the structure of the optimal controls. Sparsity can also be promoted by L^1-type and measure-space functionals; see, e.g. [2–4] and the references given there. We point out that although the desired controls are piecewise constant, problem (1.1) differs fundamentally from control problems with a total-variation-type penalty as considered in [4], since here the constants are fixed a priori. For $d = 2$ and $\alpha = \beta = 0$, problem (1.1) is a classical bang-bang control problem, where optimal controls satisfy a *generalized bang-bang-principle*, i.e., the control constraints u_1 and u_2 are attained almost everywhere outside a set where the optimal state reaches the target; see, e.g., [5–8]. The case $d = 3$ and $u_1 < u_2 = 0 < u_3$ has been treated as a "bang-sparse-bang" control problem in [1, Section 4]. In the context of time-dependent systems, controls taken pointwise in time from a discrete set of states are referred to as switching controls and have been treated in the literature mainly with respect to feedback control for ordinary differential equations and exact controllability. Regarding the former we refer to [9–11], where feedback controls and compensators are constructed that switch between a discrete set of gain operators; typically with the goal of stability of the closed loop system. In [12,13], controllability of ordinary differential equations and of the heat equation is analyzed for control actuators with switching structure.

Problem (1.1) is challenging since the penalty term is neither convex nor lower semicontinuous. We thus cannot apply the standard approach in optimal control, which consists in arguing existence of a solution via limits of a minimizing sequence and deriving necessary optimality conditions using separation theorems from convex analysis. Recall that for Fréchet-differentiable $\mathcal{F}$ and convex $\mathcal{G}$, a minimizer $\bar{u}$ of

$$\min_u \mathcal{F}(u) + \mathcal{G}(u) \tag{1.2}$$

satisfies the following necessary optimality conditions: there exists a $\bar{p} = -\mathcal{F}'(\bar{u})$ such that $\bar{p} \in \partial\mathcal{G}(\bar{u})$, which holds if and only if $\bar{u} \in \partial\mathcal{G}^*(\bar{p})$; see, e.g., [14, Proposition 4.4.4]. Here, $\mathcal{G}^*$ denotes the Fenchel conjugate of the convex functional $\mathcal{G}$, and $\partial\mathcal{G}^*$ denotes its convex subdifferential. We thus obtain the primal-dual optimality system

$$\begin{cases} -\bar{p} = \mathcal{F}'(\bar{u}), \\ \bar{u} \in \partial\mathcal{G}^*(\bar{p}). \end{cases} \tag{1.3}$$

Note that since Fenchel conjugates are always convex, this system is well-defined even for nonconvex $\mathcal{G}$, although one cannot derive it as a necessary optimality condition for minimizers of (1.2).[1] We thus follow the approach from [1], in that we show existence of a solution to (1.3) and verify that (under some conditions) it is a minimizer of (1.1). This approach is based on deriving an explicit, pointwise characterization of the subdifferential $\partial\mathcal{G}^*$, which also yields that under some assumptions on α, β, A and z, the solution will attain the values $u_1, \dots, u_d$ almost everywhere (i.e., it satisfies a *generalized multi-bang principle*). This characterization is also instrumental for the numerical solution of (1.3) using a semismooth Newton method.

This paper is organized as follows. The next section is concerned with the formal optimality system (1.3), where an explicit form is derived in Section 2.1, existence and stability of a unique solution is shown in Section 2.2, and the structure of the resulting controls – in particular, conditions for a generalized multi-bang principle – is investigated in Section 2.3. Suboptimality of controls is characterized in Section 3, and conditions for optimality are given. Section 4 addresses the computation of solutions by introducing a regularization of (1.3) for which a semismooth Newton method is applicable. Finally, Section 5 illustrates the structure of multi-bang controls with numerical examples.

2. Formal optimality system

In this section we consider the system (1.3) with

[1] This "formal convex analysis" approach should be compared to the formal Lagrangian approach for deriving explicit optimality conditions in optimal control of partial differential equations.

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