

Regularity of $p(\cdot)$ -superharmonic functions, the Kellogg property and semiregular boundary points

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Abstract

We study various boundary and inner regularity questions for $p(\cdot)$ -(super)harmonic functions in Euclidean domains. In particular, we prove the Kellogg property and introduce a classification of boundary points for $p(\cdot)$ -harmonic functions into three disjoint classes: regular, semiregular and strongly irregular points. Regular and especially semiregular points are characterized in many ways. The discussion is illustrated by examples.

Along the way, we present a removability result for bounded $p(\cdot)$ -harmonic functions and give some new characterizations of $W_0^{1,p(\cdot)}$ spaces. We also show that $p(\cdot)$ -superharmonic functions are lower semicontinuously regularized, and characterize them in terms of lower semicontinuously regularized supersolutions.

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1. Introduction

The theory of partial differential equations with nonstandard growth has been a subject of increasing interest in the last decade. Several results known for the model elliptic differential operator of nonlinear analysis, the p -Laplacian $\Delta_p := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, have been established in the variable exponent setting for the so-called $p(\cdot)$ -Laplace equation and some of its modifications. The $p(\cdot)$ -Laplace equation

$$\operatorname{div}(p(x)|\nabla u|^{p(x)-2} \nabla u) = 0$$

is the Euler–Lagrange equation for the minimization of the $p(\cdot)$ -Dirichlet integral

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$$\int_{\Omega} |\nabla u|^{p(x)} dx$$

among functions with given boundary data. Such minimization problems and equations arise for instance from applications in image processing, see Chen, Levine, and Rao [12], and in the description of electrorheological fluids, see Acerbi and Mingione [1] and Růžička [29].

Variable exponent equations have been studied, among others, in the context of interior regularity of solutions, see e.g. Acerbi and Mingione [2], Fan [14] and Henriques [23], and from the point of view of geometric properties of the solutions, see e.g. Adamowicz and Hästö [3,4]. Also, the nonlinear potential theory associated with variable exponent elliptic equations has recently attracted attention, see e.g. Harjulehto, Kinnunen, and Lukkari [19], Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16], Latvala, Lukkari, and Toivanen [25] and Lukkari [28]. For a survey of recent results in the field we refer to Harjulehto, Hästö, Lê, and Nuortio [18].

Despite the symbolic similarity to the p -Laplacian, various unexpected phenomena can occur when the exponent is a function, for instance the minimum of the $p(\cdot)$ -Dirichlet energy may not exist even in the one-dimensional case for smooth p , see [18, Section 3], and smooth functions need not be dense in the corresponding variable exponent Sobolev spaces, see the monograph by Diening, Harjulehto, Hästö, and Růžička [13, Chapter 9.2].

In this paper we address several questions regarding boundary regularity of $p(\cdot)$ -harmonic functions, i.e. the solutions of the $p(\cdot)$ -Laplace equation. Our focus is on discussing various types of boundary points and on analyzing the structure of sets of such points.

A boundary point $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega),$$

where Ω is a nonempty bounded open subset of $\mathbf{R}^n$ and Hf is the solution of the $p(\cdot)$ -Dirichlet problem with boundary values f . (See later sections for notation and precise definitions.)

Theorem 1.1 (*The Kellogg property*). *The set of all irregular boundary points has zero $p(\cdot)$ -capacity.*

The Kellogg property for variable exponents was recently obtained by Latvala, Lukkari, and Toivanen [25] using balayage and the Wiener criterion (the latter being due to Alkhutov and Krasheninnikova [5, Theorem 1.1]). Here we provide a shorter and more elementary proof, which in particular does not depend on the Wiener criterion. It is based on the ideas introduced by Björn, Björn, and Shanmugalingam [11] for their proof of the Kellogg property in metric spaces (with constant p). The proof in [11] is based on Newtonian-type Sobolev spaces, but here we have refrained from the Newtonian approach and only use the usual variable exponent Sobolev spaces. Our proof may therefore be of interest also in the constant p case, for readers who prefer to avoid Newtonian spaces.

That a boundary point is regular can be rephrased in the following way. A point $x_0 \in \partial\Omega$ is regular if the following two conditions hold:

(a) for all $f \in C(\partial\Omega)$ the limit

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) \quad \text{exists;} \tag{1.1}$$

(b) for all $f \in C(\partial\Omega)$ there is a sequence $\{y_j\}_{j=1}^\infty$ such that

$$\Omega \ni y_j \rightarrow x_0 \quad \text{and} \quad Hf(y_j) \rightarrow f(x_0), \quad \text{as } j \rightarrow \infty. \tag{1.2}$$

It turns out that for irregular boundary points *exactly one* of these two properties holds, i.e. it can never happen that both fail. This is the content of the following theorem. We say that $x_0 \in \partial\Omega$ is *semiregular* if (a) holds but not (b), and *strongly irregular* if (b) holds but not (a).

Theorem 1.2 (*Trichotomy*). *A boundary point $x_0 \in \partial\Omega$ is either regular, semiregular or strongly irregular.*

The first example (for $p = 2$) of an irregular boundary point was given by Zaremba [30] in 1911, in which he showed that the centre of a punctured disk is irregular. This is an example of a semiregular point. Shortly afterwards,

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