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Ann. I. H. Poincaré – AN 31 (2014) 1311–1335

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

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Motion of a vortex filament with axial flow in the half space

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Received 21 January 2013; received in revised form 26 September 2013; accepted 27 September 2013

Available online 10 October 2013

Abstract

We consider a nonlinear third order dispersive equation which models the motion of a vortex filament immersed in an incompressible and inviscid fluid occupying the three dimensional half space. We prove the unique solvability of initial–boundary value problems as an attempt to analyze the motion of a tornado.

Résumé

On considère une équation non linéaire dispersive de troisième ordre qui modélise le mouvement d'un filament tourbillonnaire immergé dans un fluide incompressible et non visqueux occupant le demi-espace en trois dimensions. Nous prouvons la solvabilité des problèmes aux limites comme une tentative pour analyser le mouvement d'une tornade.

Keywords: Vortex filament; Initial–boundary value problem; Nonlinear dispersive equation

1. Introduction

In this paper, we prove the unique solvability locally in time of the following initial–boundary value problems. For $\alpha < 0$,

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha\left\{\boldsymbol{x}_{sss} + \dfrac{3}{2}\boldsymbol{x}_{ss} \times (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})\right\}, & s > 0,\ t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_{ss}(0,t) = \boldsymbol{0}, & t > 0. \end{cases} \tag{1.1}$$

For $\alpha > 0$,

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha\left\{\boldsymbol{x}_{sss} + \dfrac{3}{2}\boldsymbol{x}_{ss} \times (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})\right\}, & s > 0,\ t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_s(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{x}_{ss}(0,t) = \boldsymbol{0}, & t > 0. \end{cases} \tag{1.2}$$

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http://dx.doi.org/10.1016/j.anihpc.2013.09.004

Here, $\boldsymbol{x}(s,t) = (x^1(s,t), x^2(s,t), x^3(s,t))$ is the position vector of the vortex filament parameterized by its arc length s at time t, $\times$ is the exterior product in the three dimensional Euclidean space, α is a non-zero constant that describes the magnitude of the effect of axial flow, $\boldsymbol{e}_3 = (0,0,1)$, and subscripts denote derivatives with their respective variables. Later in this paper, we will also use ∂_s and ∂_t to denote partial derivatives as well. We will refer to the equation in (1.1) and (1.2) as the vortex filament equation. We note here that the number of boundary conditions imposed changes depending on the sign of α. This is because the number of characteristic roots with a negative real part of the linearized equation, $\boldsymbol{x}_t = \alpha \boldsymbol{x}_{sss}$, changes depending on the sign of α.

Our motivation for considering (1.1) and (1.2) comes from analyzing the motion of a tornado. This paper is our humble attempt to model the motion of a tornado. While it is obvious that a vortex filament is not the same as a tornado and such modeling is questionable, many aspects of tornadoes are still unknown and we hope that our research can serve as a small step towards the complete analysis of the motion of a tornado. The boundary conditions $\boldsymbol{x}_s(0,t) = \boldsymbol{e}_3$ and $\boldsymbol{x}_{ss}(0,t) = \boldsymbol{0}$ force the filament to be perpendicular to the ground and be straight near the ground respectively. In problem (1.2), where both boundary conditions are imposed, it can be proved that if the end point of the initial vortex filament is on the ground, than the end point will stay on the ground, just as a tornado would move after it is formed.

To this end, in an earlier paper [1], the authors proved the global solvability of an initial–boundary value problem for the vortex filament equation with $\alpha = 0$, which is called the Localized Induction Equation (LIE). The LIE is an equation modeling the motion of a vortex filament without axial flow and was first proposed by Da Rios [2] in 1906 and was rediscovered by Arms and Hama [3] in 1965. Many mathematical studies have been done on the LIE since then. Nishiyama and Tani [4] proved the global solvability of the Cauchy problem. Koiso [5] also considered the Cauchy problem in a more geometrically general setting, but instead of the LIE he transformed the equation into a nonlinear Schrödinger equation via the Hasimoto transformation and proved the global solvability. In more recent years, Gutiérrez, Rivas, and Vega [6] constructed a one-parameter family of self-similar solutions of the LIE which form a corner in finite time. They further analyze the behavior of the solutions as the parameter is changed and conclude that the parameter affects the angle and shape of the final corner that is formed. Following their results, Banica and Vega [7,8] showed some asymptotic properties and the stability of the self-similar solutions obtained in [6]. Gutiérrez and Vega [9] proved the stability of self-similar solutions different from the ones treated in [7,8].

When axial flow is present (i.e. α is non-zero), many results are known for the Cauchy problem where the filament extends to spacial infinity or the filament is closed. For example, in Nishiyama and Tani [4], they proved the unique solvability globally in time in Sobolev spaces. Onodera [10,11] proved the unique solvability for a geometrically generalized equation. Segata [12] proved the unique solvability and showed the asymptotic behavior in time of the solution to the Hirota equation, given by

$$\mathrm{i}q_t = q_{xx} + \frac{1}{2}|q|^2 q + \mathrm{i}\alpha\big(q_{xxx} + |q|^2 q_x\big), \tag{1.3}$$

which can be obtained by applying the generalized Hasimoto transformation to the vortex filament equation. Since there are many results regarding the Cauchy problem for the Hirota equation and other Schrödinger type equations, it may feel more natural to see if the available theories from these results can be utilized to solve the initial–boundary value problem for (1.3), instead of considering (1.1) and (1.2) directly. Admittedly, problem (1.1) and (1.2) can be transformed into an initial–boundary value problem for the Hirota equation. But, in light of the possibility that a new boundary condition may be considered for the vortex filament equation in the future, we thought that it would be helpful to develop the analysis of the vortex filament equation itself because the Hasimoto transformation may not be applicable depending on the new boundary condition. For example, (1.1) and (1.2) model a vortex filament moving in the three dimensional half space, but if we consider a boundary that is not flat, it is nontrivial as to if we can apply the Hasimoto transformation or not, so we decided to work with the vortex filament equation directly.

For convenience, we introduce a new variable $\boldsymbol{v}(s,t) := \boldsymbol{x}_s(s,t)$ and rewrite the problems in terms of $\boldsymbol{v}$. Setting $\boldsymbol{v}_0(s) := \boldsymbol{x}_{0s}(s)$, we have for $\alpha < 0$,

$$\begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\left\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_s) + \frac{3}{2}\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss})\right\}, & s > 0,\ t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t > 0. \end{cases} \tag{1.4}$$

For $\alpha > 0$,