Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 32 (2015) 519–532

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Non-uniqueness of weak solutions to the wave map problem ☆

Klaus Widmayer

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York 10012-1185 NY, USA

Received 17 October 2013; received in revised form 4 February 2014; accepted 11 February 2014

Available online 20 February 2014

Abstract

In this note we show that weak solutions to the wave map problem in the energy-supercritical dimension 3 are not unique. On the one hand, we find weak solutions using the penalization method introduced by Shatah [12] and show that they satisfy a local energy inequality. On the other hand we build on a special harmonic map to construct a weak solution to the wave map problem, which violates this energy inequality.

Finally we establish a local weak-strong uniqueness argument in the spirit of Struwe [15] which we employ to show that one may even have a failure of uniqueness for a Cauchy problem with a stationary solution. We thus obtain a result analogous to the one of Coron [2] for the case of the heat flow of harmonic maps.

MSC: 35L05; 35L71

Keywords: Wave maps; Weak solutions; Weak-strong uniqueness

1. Introduction

The subject under consideration in this article is the following "wave map" problem. For a map $\varphi : \mathbb{R}^{1+d} \to (M, g)$ on Minkowski space $\mathbb{R}^{1+d}$ to a Riemannian manifold (M, g) we seek to find the critical points of the Lagrangian

$$\mathcal{L}(\varphi) := \frac{1}{2} \int_{\mathbb{R}^{1+d}} \langle \partial^\alpha \varphi, \partial_\alpha \varphi \rangle_g \, dx\, dt, \tag{1}$$

where we raise indices using the Minkowski metric $\eta = \operatorname{diag}\{-1, 1, \ldots, 1\}$ and repeated indices are to be summed over. The corresponding Euler–Lagrange equations yield the Cauchy problem

$$\mathcal{D}_\alpha \partial^\alpha \varphi = 0, \qquad \varphi(0) = \varphi_0, \qquad \partial_t \varphi(0) = \varphi_1, \tag{2}$$

for given (φ_0, φ_1) in appropriate function spaces and $\mathcal{D}_\alpha$ the induced covariant derivative on the pull-back tangent bundle $\varphi^{-1}TM$.

☆ Support of the National Science Foundation DMS-1001674 is gratefully acknowledged.
E-mail address: klaus@cims.nyu.edu.

http://dx.doi.org/10.1016/j.anihpc.2014.02.001

For initial data $(\varphi_0, \varphi_1) \in \dot{H}^s \times \dot{H}^{s-1}$ the scaling invariant Sobolev space has differentiability exponent $s = \frac{d}{2}$.

The well-posedness theory in the subcritical and critical cases ($s \geqslant \frac{d}{2}$) has undergone much development, which is too extensive to be summarized exhaustively here. For the case of M a sphere, a first important step beyond techniques based on Strichartz estimates was the work of Klainerman, Machedon and Selberg [9,10], where local (in time) well-posedness for regularities $s > \frac{d}{2}$ was proved using the Wave-Sobolev (or $X^{s,b}$) spaces. Subsequently Tao [16] established global well-posedness for regularities $s \geqslant \frac{d}{2}$ in dimensions $d \geqslant 2$. Finally, in [17] Tataru showed the global well-posedness for these regularities for a wide range of target geometries (in particular for all smooth compact manifolds) using adapted function spaces. These results rely on modern methods from harmonic analysis. In contrast, for dimensions $d \geqslant 4$ Shatah and Struwe [13] found a way of establishing global well-posedness using gauge theory.

In the supercritical setting $s < \frac{d}{2}$, heuristically one expects ill-posedness and is thus led to the study of weak solutions in the energy class $\dot{H}^1 \times L^2$. An advantage of these is their relatively easy constructability using penalization methods – see Shatah [12] for the case of spheres and Freire [5] for that of compact homogeneous spaces.

In a similar vein, for equivariant geometries the wave map equation reduces to a partial differential equation in $1+1$ dimensions, so that the associated elliptic problem is an ordinary differential equation and one can explicitly construct self-similar solutions that develop singularities in finite time (see Shatah [12] for first such examples). It is then not difficult to show that uniqueness of solutions may fail. More recently there have been important numerical investigations by Bizoń et al. [1] into blow-up in finite time and singularity formation for large initial data. Moreover, Donninger [4] established a result on the stability of self-similar blow-up.

A detailed characterization of self-similar solutions using Besov spaces was given by Germain in [6] and [7]. In particular, these works support the intuition that stationary weak solutions (i.e. weak harmonic maps) should be unique amongst wave maps satisfying an energy inequality, provided they minimize the Dirichlet energy. As we shall see later (in Section 2), the present article builds on this train of thought.

Furthermore, ill-posedness has been studied more comprehensively by D'Ancona and Georgiev in [3], where inter alia a non-uniqueness result with data of supercritical (but arbitrarily close to critical) regularity in dimension $d = 2$ is given.

On the other hand, in dimensions $d \geqslant 4$ Masmoudi and Planchon [11] used gauge theory to prove unconditional uniqueness[1] of solutions in the natural class $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}$.

1.1. Plan of the article

In the present article we study weak solutions of the Cauchy problem (2) in dimension $d = 3$ and $M = \mathbb{S}^2$ with the induced metric from the embedding[2] $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. This is the supercritical case for the energy norm $\dot{H}^1$: (2) is subcritical in dimension $d = 1$, critical in dimension $d = 2$ and supercritical for $d \geqslant 3$.

Our main results are Theorems 2.9 and 4.1, which assert the non-uniqueness of weak solutions by exhibiting different weak solutions to the same Cauchy problem.

More precisely, after the necessary groundwork on energy equalities in Section 2.1 we recall Shatah's method for constructing weak solutions by penalization (see Section 2.2, or [12]). This method makes use of energy conservation, and the solutions obtained thereby satisfy local (and global) energy inequalities. In Section 2.3 we contrast this by giving the construction of a weak wave map which does not satisfy such a local energy inequality (based on an example of a harmonic map which fails to minimize the Dirichlet energy, given by Hélein in [8]). We thus establish the claimed non-uniqueness of weak solutions, Theorem 2.9.

In Section 3 we prove a weak-strong uniqueness result in analogy to the one of Struwe [15]. This is interesting in its own right, but we employ it here to show that uniqueness of weak solutions can fail even in a scenario that allows for stationary solutions. This is our final result, Theorem 4.1, providing an analogy to the one of Coron [2] for the heat flow of harmonic maps.

It is worth noting that our strategy of proof applies in more general situations: On the one hand, whenever there is a weak harmonic map whose stress–energy tensor does not vanish we can construct a weak wave map that violates the energy inequality – in particular, this is the case if the harmonic map is not energy minimizing. On the other hand, if

---

1 I.e. uniqueness without further assumptions of boundedness of higher order Lebesgue norms.

2 We adopt this external point of view for the rest of this article, in particular also for the formulation of the equation.