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Nonlinear scalar field equations: Existence of a positive solution with infinitely many bumps ☆

Giovanna Cerami [a], Donato Passaseo [b], Sergio Solimini [a,*]

[a] *Politecnico di Bari, Dipartimento di Meccanica, Matematica e Management, Via Orabona 4, 70125 Bari, Italy*
[b] *Università del Salento, Dipartimento di Matematica e Fisica, Ex Collegio Fiorini, Via per Arnesano, 73047 Monteroni di Lecce (LE), Italy*

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Abstract

In this paper we consider the equation

$$(E)\quad -\Delta u + a(x)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 2$, $p > 1$, $p < 2^* - 1 = \frac{N+2}{N-2}$, if $N \geqslant 3$. During last thirty years the question of the existence and multiplicity of solutions to (E) has been widely investigated mostly under symmetry assumptions on a. The aim of this paper is to show that, differently from those found under symmetry assumption, the solutions found in [6] admit a limit configuration and so (E) also admits a positive solution of infinite energy having infinitely many 'bumps'.

Résumé

Dans ce papier nous considérons l'équation

$$(E)\quad -\Delta u + a(x)u = |u|^{p-1}u \quad \text{en } \mathbb{R}^N,$$

où $N \geqslant 2$, $p > 1$, $p < 2^* - 1 = \frac{N+2}{N-2}$, si $N \geqslant 3$. Pendant les trente dernières années la question de l'existence et de la multiplicité de solutions d'(E) a été largement examinée surtout conformément aux suppositions de symétrie sur a. Le but de ce papier est de montrer que, différemment de ceux trouvés conformément à la supposition de symétrie, les solutions trouvées dans [6] admettent une configuration de limite et donc (E) admet aussi une solution positive d'énergie infinie ayant une infinité de 'bumps'.

MSC: 35J20; 35J60; 35Q55

Keywords: Variational methods; Solutions with infinitely many bumps; Schrödinger equation

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* Corresponding author.
E-mail addresses: cerami@poliba.it (G. Cerami), donato.passaseo@unisalento.it (D. Passaseo), solimini@poliba.it (S. Solimini).

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1. Introduction

In this paper we consider the equation

$$(E)\quad -\Delta u + a(x)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 2$, $p > 1$, $p < 2^* - 1 = \frac{N+2}{N-2}$, if $N \geqslant 3$, and the potential $a(x)$ is a positive function that is not required to enjoy symmetry.

During the past years there has been a considerable amount of research on this kind of questions; the interest comes, essentially, from two reasons: their specific mathematical difficulties, that make them challenging to the researchers, and, moreover, the fact that equations as (E) arise naturally in several branches of mathematical physics. Indeed the solutions of (E) can be seen as stationary states (corresponding to solitary waves) in nonlinear equations of the Klein–Gordon type

$$\frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi + \left(a(x) + \omega^2\right)\varphi - |\varphi|^{p-2}\varphi = 0 \tag{1}$$

and of Schrödinger type

$$i\frac{\partial \varphi}{\partial t} - \Delta\varphi + \left(a(x) + \omega^2\right)\varphi - |\varphi|^{p-2}\varphi = 0 \tag{2}$$

(where $\varphi = \varphi(t,x)$) is a complex function defined on $\mathbb{R} \times \mathbb{R}^N$.

Let us consider, for instance, Eq. (1): it corresponds to the Lagrangian density

$$\mathcal{L}(\varphi) = -\frac{1}{2}|\varphi_t|^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\left(a(x) + \omega^2\right)|\varphi|^2 - \frac{1}{p}|\varphi|^p.$$

Thus, looking for a solitary wave, of the standing wave form, means searching solutions $\varphi(x,t) = e^{i\omega t}u(x)$, with $u : \mathbb{R}^N \to \mathbb{R}$, hence one is led exactly to the equation considered in (E). Analogously searching for stationary states of (2) leads again to (E).

Furthermore, we recall that, besides the above mentioned problems, equations like (E), which are also called Euclidean scalar field equations, appear in several other context of physics: nonlinear optics, laser propagations, constructive field theory, etc.

During last thirty years the question of the existence and multiplicity of solutions to (E) has been widely investigated and most results have been obtained under symmetry assumptions on a. However, some considerable progress has been performed also in the case in which $a(x)$ is not required to fulfill symmetry properties. We just mention that, formerly, the existence of a positive solution has been shown in [7,1,2], while the existence of infinitely many changing sign solutions has been proven in [5].

We refer the interested reader either to [4] or to [6] for a more detailed description of the development of the researches as well as a quite exhaustive list of references.

Very recently, an answer to the question of the existence of infinitely many positive solutions to (E) has been given in [6] where the following result has been proved:

Theorem 1. *Let assumptions*

(h_1) $a(x) \to a_\infty > 0$ *as* $|x| \to \infty$,
(h_2) $a(x) \geqslant a_0 > 0$, $\forall x \in \mathbb{R}^N$,
(h_3) $a \in L^{N/2}_{loc}(\mathbb{R}^N)$,
(h_4) $\exists \bar\eta \in (0, \sqrt{a_\infty})$: $\lim_{|x|\to+\infty}(a(x) - a_\infty)e^{\bar\eta|x|} = +\infty$

be satisfied.

Then there exists a positive constant, $\mathcal{A} = \mathcal{A}(N, \bar\eta, a_0, a_\infty) \in \mathbb{R}$, *such that, when*

$$\left|a(x) - a_\infty\right|_{N/2,loc} := \sup_{y\in\mathbb{R}^N}\left|a(x) - a_\infty\right|_{L^{N/2}(B_1(y))} < \mathcal{A},$$

equation (E) *has infinitely many positive solutions belonging to* $H^1(\mathbb{R}^N)$.