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Time fluctuations in a population model of adaptive dynamics

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Abstract

We study the dynamics of phenotypically structured populations in environments with fluctuations. In particular, using novel arguments from the theories of Hamilton–Jacobi equations with constraints and homogenization, we obtain results about the evolution of populations in environments with time oscillations, the development of concentrations in the form of Dirac masses, the location of the dominant traits and their evolution in time. Such questions have already been studied in time homogeneous environments. More precisely we consider the dynamics of a phenotypically structured population in a changing environment under mutations and competition for a single resource. The mathematical model is a non-local parabolic equation with a periodic in time reaction term. We study the asymptotic behavior of the solutions in the limit of small diffusion and fast reaction. Under concavity assumptions on the reaction term, we prove that the solution converges to a Dirac mass whose evolution in time is driven by a Hamilton–Jacobi equation with constraint and an effective growth/death rate which is derived as a homogenization limit. We also prove that, after long-time, the population concentrates on a trait where the maximum of an effective growth rate is attained. Finally we provide an example showing that the time oscillations may lead to a strict increase of the asymptotic population size.

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1. Introduction

Phenotypically structured populations can be modeled using non-local Lotka–Volterra equations, which have the property that, in the small mutations limit, the solutions concentrate on one or several evolving in time Dirac masses. A recently developed mathematical approach, which uses Hamilton–Jacobi equations with constraint, allows us to understand the behavior of the solutions in constant environments [5,12,1,10].

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Since stochastic and periodic modulations are important for the modeling [9,14,15,8,16], a natural and relevant question is whether it is possible to further develop the theory to models with time fluctuating environments.

In this note we consider an environment which varies periodically in time in order, for instance, to take into account the effect of seasonal variations in the dynamics, and we study the asymptotic properties of the initial value problem

$$\begin{cases} \varepsilon n_{\varepsilon,t} = n_\varepsilon R\left(x, \frac{t}{\varepsilon}, I_\varepsilon(t)\right) + \varepsilon^2 \Delta n_\varepsilon \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ n_\varepsilon(\cdot,0) = n_{0,\varepsilon} \quad \text{in } \mathbb{R}^N, \\ I_\varepsilon(t) := \int\limits_{\mathbb{R}^N} \psi(x) n_\varepsilon(x,t)\,dx, \end{cases} \tag{1}$$

where

$$R : \mathbb{R}^N \times \mathbb{R} \times [0,\infty) \to \mathbb{R} \quad \text{is smooth and 1-periodic in its second argument.} \tag{2}$$

The population is structured by phenotypical traits $x \in \mathbb{R}^N$ with density $n_\varepsilon(x,t)$ at time t. It is assumed that there exists a single type of resource which is consumed by each individual trait x at a rate $\psi(x)$; $I_\varepsilon(t)$ is then the total consumption of the population. The mutations and the growth rate are represented respectively by the Laplacian term and R. The novelty is the periodic in time dependence of the growth rate R. The small coefficient ε is used to consider only rare mutations and to rescale time in order to study a time scale much larger than the generation one.

To ensure the survival and the boundedness of the population we assume that R takes positive values for "small enough populations" and negative values for "large enough populations", i.e., there exists a value $I_M > 0$ such that

$$\max_{0 \leqslant s \leqslant 1,\, x \in \mathbb{R}^N} R(x,s,I_M) = 0 \quad \text{and} \quad \mathcal{X} := \left\{ x \in \mathbb{R}^N,\ \int_0^1 R(x,s,0)\,ds > 0 \right\} \neq \emptyset. \tag{3}$$

In addition the growth rate R satisfies, for some positive constants K_i, $i = 1,\dots,7$, and all $(x,s,I) \in \mathbb{R}^N \times \mathbb{R} \times [0, I_M]$ and $A > 0$, the following concavity and decay assumptions:

$$-2K_1 \leqslant D_x^2 R(x,s,I) \leqslant -2K_2, \qquad -K_3 - K_1|x|^2 \leqslant R(x,s,I) \leqslant K_4 - K_2|x|^2, \tag{4}$$

$$-K_5 \leqslant D_I R(x,s,I) \leqslant -K_6, \tag{5}$$

$$D_x^3 R \in L^\infty\left(\mathbb{R}^N \times (0,1) \times [0,A]\right) \quad \text{and} \quad \left|D_{x,I}^2 R\right| \leqslant K_7. \tag{6}$$

The "uptake coefficient" $\psi : \mathbb{R}^N \to \mathbb{R}$ must be regular and bounded from above and below, i.e., there exist positive constants ψ_m, ψ_M and K_8 such that

$$0 < \psi_m \leqslant \psi \leqslant \psi_M \quad \text{and} \quad \|\psi\|_{C^2} \leqslant K_8. \tag{7}$$

We also assume that the initial datum is "asymptotically monomorphic", i.e., it is close to a Dirac mass in the sense that there exist $x^0 \in \mathcal{X}$, $\rho^0 > 0$ and a smooth $u_\varepsilon^0 : \mathbb{R}^N \to \mathbb{R}$ such that

$$n_{0,\varepsilon} = e^{u_\varepsilon^0/\varepsilon} \quad \text{and, as } \varepsilon \to 0, \tag{8}$$

$$n_\varepsilon(\cdot,0) \underset{\varepsilon\to 0}{\longrightarrow} \varrho^0 \delta\left(\cdot - x^0\right) \quad \text{weakly in the sense of measures.} \tag{9}$$

In addition there exist constants $L_i > 0$, $i = 1,\dots,4$, and a smooth $u^0 : \mathbb{R}^N \to \mathbb{R}$ such that, for all $x \in \mathbb{R}^N$,

$$-2L_1 I \leqslant D_x^2 u_\varepsilon^0 \leqslant -2L_2 I, \qquad -L_3 - L_1|x|^2 \leqslant u_\varepsilon^0(x) \leqslant L_4 - L_2|x|^2, \qquad \max_{x \in \mathbb{R}^N} u^0(x) = 0 = u^0\left(x^0\right) \tag{10}$$

and, as $\varepsilon \to 0$,

$$u_\varepsilon^0 \longrightarrow u^0 \quad \text{locally uniformly in } \mathbb{R}^N. \tag{11}$$

Finally, in order to ensure the same control on $D^2 u$ as for $D^2 u^0$, it is necessary to impose the following compatibility relation on the parameters in the initial data and in the growth rate R:

$$4L_2^2 \leqslant K_2 \leqslant K_1 \leqslant 4L_1^2. \tag{12}$$