Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 32 (2015) 183–200

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

A new method for large time behavior of degenerate viscous Hamilton–Jacobi equations with convex Hamiltonians

Filippo Cagnetti [a], Diogo Gomes [a,b], Hiroyoshi Mitake [c], Hung V. Tran [d,*]

[a] *Center for Mathematical Analysis, Geometry, and Dynamical Systems, Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal*

[b] *King Abdullah University of Science and Technology (KAUST), CSMSE Division, Thuwal 23955-6900, Saudi Arabia*

[c] *Department of Applied Mathematics, Faculty of Science, Fukuoka University, Fukuoka 814-0180, Japan*

[d] *Department of Mathematics, The University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA*

Received 22 October 2013; accepted 29 October 2013

Available online 16 November 2013

Abstract

We investigate large-time asymptotics for viscous Hamilton–Jacobi equations with possibly degenerate diffusion terms. We establish new results on the convergence, which are the first general ones concerning equations which are neither uniformly parabolic nor first order. Our method is based on the nonlinear adjoint method and the derivation of new estimates on long time averaging effects. It also extends to the case of weakly coupled systems.

MSC: 35B40; 35F55; 49L25

Keywords: Large-time behavior; Hamilton–Jacobi equations; Degenerate parabolic equations; Nonlinear adjoint methods; Viscosity solutions

1. Introduction

In this paper we obtain new results on the study of the large time behavior of Hamilton–Jacobi equations with possibly degenerate diffusion terms

$$u_t + H(x, Du) = \operatorname{tr}\big(A(x)D^2u\big) \quad \text{in } \mathbb{T}^n \times (0, \infty), \tag{1.1}$$

where $\mathbb{T}^n$ is the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$. Here Du, D^2u are the (spatial) gradient and Hessian of the real-valued unknown function u defined on $\mathbb{T}^n \times [0, \infty)$. The functions $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ and $A : \mathbb{T}^n \to \mathbb{M}^{n\times n}_{\mathrm{sym}}$ are the Hamiltonian and the diffusion matrix, respectively, where $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ is the set of $n \times n$ real symmetric matrices. The basic hypotheses that we require are that H is uniformly convex in the second variable, and A is nonnegative definite.

* Corresponding author.
E-mail addresses: cagnetti@math.ist.utl.pt (F. Cagnetti), dgomes@math.ist.utl.pt (D. Gomes), mitake@math.sci.fukuoka-u.ac.jp (H. Mitake), hung@math.uchicago.edu (H.V. Tran).

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.10.005

Our goal in this paper is to study the large time behavior of viscosity solutions of (1.1). Namely, we prove that

$$\|u(\cdot,t)-(v-ct)\|_{L^\infty(\mathbb{T}^n)} \to 0 \quad \text{as } t\to\infty, \tag{1.2}$$

where (v,c) is a solution of the *ergodic problem*

$$H(x,Dv) = \operatorname{tr}\big(A(x)D^2v\big) + c \quad \text{in } \mathbb{T}^n. \tag{1.3}$$

In view of the quadratic or superquadratic growth of the Hamiltonian, there exists a unique constant $c \in \mathbb{R}$ such that (1.3) holds true for some $v \in C(\mathbb{T}^n)$ in the viscosity sense. We notice that in the uniformly parabolic case (A is positive definite), v is unique up to additive constants. It is however typically the case that v is not unique even up to additive constants when A is degenerate, which makes the convergence (1.2) delicate and hard to be achieved. We will state clearly the existence result of (1.3), which itself is important, in Section 2.

It is worth emphasizing here that the study of the large-time asymptotics for this type of equations was only available in the literature for the uniformly parabolic case and for the first order case. There was no results on the large-time asymptotics for (1.1) with possibly degenerate diffusion terms up to now as far as the authors know.

In the last decade, a number of authors have studied extensively the large time behavior of solutions of (first order) Hamilton–Jacobi equations (i.e., (1.1) with $A \equiv 0$), where H is coercive. Several convergence results have been established. The first general theorem in this direction was proven by Namah and Roquejoffre in [18], under the assumptions: $p \mapsto H(x,p)$ is convex, $H(x,p) \geqslant H(x,0)$ for all $(x,p) \in \mathbb{T}^n \times \mathbb{R}^n$, and $\max_{x\in\mathbb{T}^n} H(x,0) = 0$. Fathi then gave a breakthrough in this area in [10] by using a dynamical systems approach from the weak KAM theory. Contrary to [18], the results of [10] use uniform convexity and smoothness assumptions on the Hamiltonian but do not require any condition on the structure above. These rely on a deep understanding of the dynamical structure of the solutions and of the corresponding ergodic problem. See also the paper of Fathi and Siconolfi [11] for a beautiful characterization of the Aubry set. Afterwards, Davini and Siconolfi in [7] and Ishii in [12] refined and generalized the approach of Fathi, and studied the asymptotic problem for Hamilton–Jacobi equations on $\mathbb{T}^n$ and on the whole n-dimensional Euclidean space, respectively. Besides, Barles and Souganidis [2] obtained additional results, for possibly non-convex Hamiltonians, by using a PDE method in the context of viscosity solutions. Barles, Ishii and Mitake [1] simplified the ideas in [2] and presented the most general assumptions (up to now). In general, these methods are based crucially on delicate stability results of extremal curves in the context of the dynamical approach in light of the finite speed of propagation, and of solutions for time large in the context of the PDE approach. It is also important to point out that the PDE approach in [2,1] does not work with the presence of any second order terms.

In the uniformly parabolic setting (i.e., A uniformly positive definite), Barles and Souganidis [3] proved the long-time convergence of solutions. Their proof relies on a completely distinct set of ideas from the ones used in the first order case as the associated ergodic problem has a simpler structure. Indeed, the strong maximum principle holds, the ergodic problem has a unique solution up to constants. The proof for the large-time convergence in [3] strongly depends on this fact.

It is clear that all the methods aforementioned (for both the cases $A \equiv 0$ and A uniformly positive definite) are not applicable for the general degenerate viscous cases because of the presence of the second order terms and the lack of both the finite speed of propagation as well as the strong comparison principle. We briefly describe the key ideas on establishing (1.2) in Subsection 1.1. Here the nonlinear adjoint method, which was introduced by Evans in [8], plays the essential role in our analysis. Our main results are stated in Subsection 1.2.

1.1. Key ideas

Let us now briefly describe the key ideas on establishing (1.2). Without loss of generality, we may assume the ergodic constant is 0 henceforth. In order to understand the limit as $t\to\infty$, we introduce a rescaled problem. For $\varepsilon>0$, set $u^\varepsilon(x,t) = u(x,t/\varepsilon)$. Then $(u^\varepsilon)_t(x,t) = \varepsilon^{-1}u_t(x,t/\varepsilon)$, $Du^\varepsilon(x,t) = Du(x,t/\varepsilon)$, and u^ε solves

$$\begin{cases} \varepsilon u^\varepsilon_t + H\big(x,Du^\varepsilon\big) = \operatorname{tr}\big(A(x)D^2u^\varepsilon\big) & \text{in } \mathbb{T}^n\times(0,\infty),\\ u^\varepsilon(x,0) = u_0(x), & \text{on } \mathbb{T}^n. \end{cases}$$

By this rescaling, $u^\varepsilon(x,1) = u(x,1/\varepsilon)$ and we can easily see that to prove (1.2) is equivalent to prove that

$$\|u^\varepsilon(\cdot,1) - v\|_{L^\infty(\mathbb{T}^n)} \to 0 \quad \text{as } \varepsilon\to 0.$$