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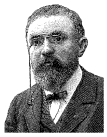
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Compactness and bubble analysis for 1/2-harmonic maps

Francesca Da Lio

Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland

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Abstract

In this paper we study compactness and quantization properties of sequences of $1/2$-harmonic maps $u_k : \mathbb{R} \to \mathcal{S}^{m-1}$ such that $\|u_k\|_{\dot{H}^{1/2}(\mathbb{R},\mathcal{S}^{m-1})} \leqslant C$. More precisely we show that there exist a weak $1/2$-harmonic map $u_\infty : \mathbb{R} \to \mathcal{S}^{m-1}$, a finite and possible empty set $\{a_1, \dots, a_\ell\} \subset \mathbb{R}$ such that up to subsequences

$$\left|(-\Delta)^{1/4} u_k\right|^2 dx \rightharpoonup \left|(-\Delta)^{1/4} u_\infty\right|^2 dx + \sum_{i=1}^{\ell} \lambda_i \delta_{a_i}, \quad \text{in Radon measure,}$$

as $k \to +\infty$, with $\lambda_i \geqslant 0$.

The convergence of u_k to u_∞ is strong in $\dot{W}^{1/2,p}_{loc}(\mathbb{R} \setminus \{a_1, \dots, a_\ell\})$, for every $p \geqslant 1$. We quantify the loss of energy in the weak convergence and we show that in the case of non-constant $1/2$-harmonic maps with values in $\mathcal{S}^1$ one has $\lambda_i = 2\pi n_i$, with n_i a positive integer.

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1. Introduction

In the paper [9] Rivière and the author started the investigation of the following 1-dimensional quadratic Lagrangian

$$L(u) = \int_{\mathbb{R}} \left|(-\Delta)^{1/4} u(x)\right|^2 dx, \tag{1}$$

where $u : \mathbb{R} \to \mathcal{N}$, $\mathcal{N}$ is a smooth k-dimensional sub-manifold of $\mathbb{R}^m$ which is at least C^2, compact and without boundary. We observe that (1) is a simple model of Lagrangian which is invariant under the trace of conformal maps that keep invariant the half-space $\mathbb{R}^2_+$: the Möbius group.

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Precisely let $\phi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ in $W^{1,2}(\mathbb{R}^2_+, \mathbb{R}^2_+)$ be a conformal map of degree 1, i.e. it satisfies

$$\begin{cases} \left|\dfrac{\partial \phi}{\partial x}\right| = \left|\dfrac{\partial \phi}{\partial y}\right|, \\ \left\langle \dfrac{\partial \phi}{\partial x}, \dfrac{\partial \phi}{\partial y}\right\rangle = 0, \\ \det \nabla\phi \geqslant 0 \quad \text{and} \quad \nabla\phi \neq 0. \end{cases} \tag{2}$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^m$.

We denote by $\tilde{\phi}$ the restriction of ϕ to $\mathbb{R}$. Then we have $L(u \circ \tilde{\phi}) = L(u)$.

Moreover $L(u)$ in (1) coincides with the semi-norm $\|u\|^2_{\dot{H}^{1/2}(\mathbb{R})}$ and the following identity holds

$$\int_{\mathbb{R}} \left|(-\Delta)^{1/4} u(x)\right|^2 dx = \inf\left\{ \int_{\mathbb{R}^2_+} |\nabla \tilde{u}|^2 \, dx \colon \tilde{u} \in W^{1,2}\left(\mathbb{R}^2, \mathbb{R}^m\right), \text{ trace } \tilde{u} = u \right\}. \tag{3}$$

The Lagrangian L extends to maps u in the following function space

$$\dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) = \left\{ u \in \dot{H}^{1/2}\left(\mathbb{R}, \mathbb{R}^m\right) \colon u(x) \in \mathcal{N}, \text{ a.e.} \right\}.$$

The operator $(-\Delta)^{1/4}$ on $\mathbb{R}$ is defined by means of the Fourier transform as follows

$$\widehat{(-\Delta)^{1/4} u} = |\xi|^{1/2} \hat{u}$$

(given a function f, $\hat{f}$ denotes the Fourier transform of f).

We denote by $\pi_{\mathcal{N}}$ the orthogonal projection from $\mathbb{R}^m$ onto $\mathcal{N}$ which happens to be a C^ℓ map in a sufficiently small neighborhood of $\mathcal{N}$ if $\mathcal{N}$ is assumed to be $C^{\ell+1}$. We now introduce the notion of 1/2-harmonic map into a manifold.

Definition 1.1. A map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is called a weak 1/2-harmonic map into $\mathcal{N}$ if for any $\phi \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ there holds

$$\frac{d}{dt} L\left(\pi_{\mathcal{N}}(u + t\phi)\right)_{|t=0} = 0.$$

In short we say that a weak 1/2-harmonic map is a *critical point of* L *in* $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ *for perturbations in the target*.

We next give some geometric motivations related to the study of the problem (1).

First of all variational problems of the form (1) appear as a simplified model of renormalization area in hyperbolic spaces, see for instance [2]. There are also some geometric connections which are being investigated in the paper [11] between 1/2-harmonic maps and the so-called free boundary sub-manifolds and optimization problems of eigenvalues. With this regards we refer the reader also to the papers [14,15]. Finally 1/2-harmonic maps into the circle $\mathcal{S}^1$ might appear for instance in the asymptotics of equations in phase-field theory for fractional reaction–diffusion such as

$$\epsilon^2 (-\Delta)^{1/2} u + u\left(1 - |u|^2\right) = 0,$$

where u is a complex-valued "wave function".

In this paper we consider the case $\mathcal{N} = \mathcal{S}^{m-1}$. It can be shown (see [9]) that every weak 1/2-harmonic map satisfies the following Euler–Lagrange equation

$$(-\Delta)^{1/2} u \wedge u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{4}$$

One of the main achievements of the paper [9] is the rewriting of Eq. (4) in a more "tractable" way in order to be able to investigate regularity and compactness property of weak 1/2-harmonic maps. Precisely in [9] the following two results have been proved: