Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 31 (2014) 877–897

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Reconstruction of inhomogeneous conductivities via the concept of generalized polarization tensors ☆

Habib Ammari [a], Youjun Deng [a,*], Hyeonbae Kang [b], Hyundae Lee [b]

[a] *Department of Mathematics and Applications, École Normale Supérieure, 45 Rue d'Ulm, 75005 Paris, France*
[b] *Department of Mathematics, Inha University, Incheon 402-751, Republic of Korea*

Received 19 November 2012; received in revised form 30 July 2013; accepted 31 July 2013

Available online 30 August 2013

Abstract

This paper extends the concept of generalized polarization tensors (GPTs), which was previously defined for inclusions with homogeneous conductivities, to inhomogeneous conductivity inclusions. We begin by giving two slightly different but equivalent definitions of the GPTs for inhomogeneous inclusions. We then show that, as in the homogeneous case, the GPTs are the basic building blocks for the far-field expansion of the voltage in the presence of the conductivity inclusion. Relating the GPTs to the Neumann-to-Dirichlet (NtD) map, it follows that the full knowledge of the GPTs allows unique determination of the conductivity distribution. Furthermore, we show important properties of the the GPTs, such as symmetry and positivity, and derive bounds satisfied by their harmonic sums. We also compute the sensitivity of the GPTs with respect to changes in the conductivity distribution and propose an algorithm for reconstructing conductivity distributions from their GPTs. This provides a new strategy for solving the highly nonlinear and ill-posed inverse conductivity problem. We demonstrate the viability of the proposed algorithm by preforming a sensitivity analysis and giving some numerical examples.

MSC: 35R30; 35C20

Keywords: Generalized polarization tensors; Inhomogeneous conductivity; Neumann-to-Dirichlet map; Asymptotic expansion; Inverse conductivity problem

1. Introduction

There are several geometric and physical quantities associated with shapes such as eigenvalues and capacities [34]. The concept of the generalized polarization tensors (GPTs) is one of them. The notion appears naturally when we describe the perturbation of the electrical potential due to the presence of inclusions whose material parameter (conductivity) is different from that of the background.

☆ This work was supported by the ERC Advanced Grant Project MULTIMOD-267184 and the Ministry of Education, Sciences and Technology of Korea through NRF grants Nos. 2009-0090250, 2010-0004091 and 2010-0017532.

* Corresponding author.
E-mail addresses: habib.ammari@ens.fr (H. Ammari), deng@dma.ens.fr (Y. Deng), hbkang@inha.ac.kr (H. Kang), hdlee@inha.ac.kr (H. Lee).

http://dx.doi.org/10.1016/j.anihpc.2013.07.008

To mathematically introduce the concept of GPTs, we consider the conductivity problem in $\mathbb{R}^d$, $d = 2, 3$:

$$\begin{cases} \nabla \cdot \big(\chi\big(\mathbb{R}^d \setminus \overline{\Omega}\big) + k\chi(\Omega)\big)\nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - h(x) = O\big(|x|^{1-d}\big) & \text{as } |x| \to \infty. \end{cases} \tag{1.1}$$

Here, Ω is the inclusion embedded in $\mathbb{R}^d$ with a Lipschitz boundary, $\chi(\Omega)$ (resp. $\chi(\mathbb{R}^d \setminus \overline{\Omega})$) is the characteristic function of Ω (resp. $\mathbb{R}^d \setminus \overline{\Omega}$), the positive constant k is the conductivity of the inclusion which is supposed to be different from the background conductivity 1, h is a harmonic function in $\mathbb{R}^d$ representing the background electrical potential, and the solution u to the problem represents the perturbed electrical potential. The perturbation $u - h$ due to the presence of the conductivity inclusion Ω admits the following asymptotic expansion as $|x| \to \infty$:

$$u(x) - h(x) = \sum_{|\alpha|, |\beta| \geqslant 1} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial^\alpha h(0) M_{\alpha\beta}(k, \Omega) \partial^\beta \Gamma(x), \tag{1.2}$$

where Γ is the fundamental solution of the Laplacian (see, for example, [7,9]). The building blocks $M_{\alpha\beta}(k, \Omega)$ for the asymptotic expansion (1.2) are called the GPTs. Note that the GPTs $M_{\alpha\beta}(k, \Omega)$ can be reconstructed from the far-field measurements of u by a least-squares method. A stability analysis of the reconstruction is provided in [1]. On the other hand, it is shown in [2] that in the full-view case, the reconstruction problem of GPTs from boundary data has the remarkable property that low order GPTs are not affected by the error caused by the instability of higher-orders in the presence of measurement noise.

The GPTs carry geometric information about the inclusion. For example, the inverse GPT problem holds to be true, namely, the whole set of GPTs, $\{M_{\alpha\beta}(k, \Omega)\colon |\alpha|, |\beta| \geqslant 1\}$, determines k and Ω uniquely [6]. The leading order GPT (called the polarization tensor (PT)), $\{M_{\alpha\beta}(k, \Omega)\colon |\alpha|, |\beta| = 1\}$, provides the equivalent ellipse (ellipsoid) which represents overall property of the inclusion [11,20]. Moreover, there are important analytical and numerical studies which show that finer details of the shape can be recovered using higher-order GPTs [14,4]. The GPTs even carry topology information of the inclusion [4]. It is also worth mentioning that an efficient algorithm for computing the GPTs is presented in [21].

The notion of GPTs appears in various contexts such as asymptotic models of dilute composites (*cf.* [30,32,13]), low-frequency asymptotics of waves [24], potential theory related to certain questions arising in hydrodynamics [34], biomedical imaging of small inclusions (see [10] and the references therein), reconstructing small inclusions [27,11,20], and shape description [4]. Recently the concept of GPTs finds another promising application to cloaking and electromagnetic and acoustic invisibility. It is shown that the near-cloaking effect of [29] can be dramatically improved by using multi-layered structures whose GPTs vanish up to a certain order [12].

As far as we know, the GPTs have been introduced only for inclusions with homogeneous conductivities or layers with constant conductivities. It is the purpose of this paper to extend the notion of GPTs to inclusions with inhomogeneous conductivities and use this new concept for solving the inverse conductivity problem. We first introduce the GPTs for inhomogeneous inclusions and show that exactly the same kind of far-field asymptotic formula as (1.2) holds. We also prove important properties of the GPTs such as unique determination of Neumann-to-Dirichlet map, symmetry, and positivity. We then provide a sensitivity analysis of the GPTs with respect to changes in the conductivity distribution. We finally propose a minimization algorithm for reconstructing an inhomogeneous conductivity distribution from its high-order GPTs. We carry out a resolution and stability analysis for this reconstruction problem in the linearized case and present numerical examples to show its viability.

The paper is organized as follows. In Section 2 we introduce the GPTs for inhomogeneous conductivity inclusions and prove that they are the building blocks of the far-field expansion of the potential. Section 3 is devoted to the derivation of integral representations of the GPTs. We also establish a relation between the GPTs and the NtD map. In Section 4 we prove important properties of symmetry and positivity of the GPTs and obtain bounds satisfied by their harmonic sums. In Section 5 we perform a sensitivity analysis of the GPTs with respect to the conductivity distribution. We also show that in the linearized case, high-order GPTs capture high-frequency oscillations of the conductivity. In Section 6, we present an algorithm for reconstructing inhomogeneous conductivity distributions from their high-order GPTs. The algorithm is based on minimizing the discrepancy between the computed and measured GPTs.