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Estimates on fractional higher derivatives of weak solutions for the Navier–Stokes equations

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Abstract

We study weak solutions of the 3D Navier–Stokes equations with L^2 initial data. We prove that $\nabla^\alpha u$ is locally integrable in space–time for any real α such that $1 < \alpha < 3$. Up to now, only the second derivative $\nabla^2 u$ was known to be locally integrable by standard parabolic regularization. We also present sharp estimates of those quantities in weak-$L^{4/(\alpha+1)}_{loc}$. These estimates depend only on the L^2-norm of the initial data and on the domain of integration. Moreover, they are valid even for $\alpha \geqslant 3$ as long as u is smooth. The proof uses a standard approximation of Navier–Stokes from Leray and blow-up techniques. The local study is based on De Giorgi techniques with a new pressure decomposition. To handle the non-locality of fractional Laplacians, Hardy space and Maximal functions are introduced.
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1. Introduction and the main result

In this paper, any derivative signs (∇, Δ, $(-\Delta)^{\alpha/2}$, D, ∂ and etc.) denote derivatives in the only space variable $x \in \mathbb{R}^3$ unless the time variable $t \in \mathbb{R}$ is clearly specified. We study the 3D Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla P - \Delta u = 0 \quad \text{and}$$
$$\operatorname{div} u = 0, \quad t \in (0, \infty),\ x \in \mathbb{R}^3, \tag{1}$$

with L^2 initial data

$$u_0 \in L^2(\mathbb{R}^3), \quad \operatorname{div} u_0 = 0. \tag{2}$$

The problem of global regularity of weak solutions for the 3D Navier–Stokes equations has a long history. Leray [27] 1930s and Hopf [22] 1950s proved the existence of a global-time weak solution for any given L^2 initial

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data. Such Leray–Hopf weak solutions are weak solutions u of (1) lying in the functional class $L^\infty(0,\infty;L^2(\mathbb{R}^3))\cap L^2(0,\infty;H^1(\mathbb{R}^3))$ and satisfying the following global energy inequality:

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)}+2\|\nabla u\|^2_{L^2(0,t;L^2(\mathbb{R}^3))}\leqslant\|u_0\|^2_{L^2(\mathbb{R}^3)}\quad\text{for a.e. } 0<t<\infty.$$

Until now, regularity and uniqueness of such weak solutions are generally open.

Instead, many criteria which ensure regularity of weak solutions have been developed. Among them, the most famous one is Ladyženskaja–Prodi–Serrin Criteria [24,30,36], which says: if $u\in L^p((0,T);L^q(\mathbb{R}^3))$ for some p and q satisfying $\frac{2}{p}+\frac{3}{q}=1$ and $p<\infty$, then it is regular. Recently, the limiting case $p=\infty$ was established in the paper of Escauriaza, Seregin and Šverák [16]. Similar criteria exist with various conditions on derivatives of velocity, vorticity, or pressure (see Beale, Kato and Majda [1], Beirão da Veiga [2] and Berselli and Galdi [4]). Also, many other conditions exist (e.g. see Cheskidov and Shvydkoy [10], Chan [9] and [5]).

On the other hand, many efforts have been devoted to the estimation of the size of the possible singular set where singularities may occur. This approach has been initiated by Scheffer [33]. Then, Caffarelli, Kohn and Nirenberg [6] improved the result and showed that possible singular sets have zero Hausdorff measure of one dimension for certain class of weak solutions (suitable weak solutions) satisfying the following additional inequality

$$\partial_t\frac{|u|^2}{2}+\operatorname{div}\left(u\frac{|u|^2}{2}\right)+\operatorname{div}(uP)+|\nabla u|^2-\Delta\frac{|u|^2}{2}\leqslant 0 \tag{3}$$

in the sense of distribution. There are many other proofs of this fact (e.g. see Lin [28], [42] and Wolf [43]). Similar criteria for interior points with other quantities can be found in many places (e.g. see Struwe [40], Gustafson, Kang and Tsai [21], Seregin [35] and Chae, Kang and Lee [8]). Also, Robinson and Sadowski [31] and Kukavica [23] studied box-counting dimensions of singular sets.

In this paper, we consider space–time $L^p_{(t,x)}=L^p_tL^p_x$-estimates of higher derivatives for weak solutions assuming only L^2 initial data. The estimate $\nabla u\in L^2((0,\infty)\times\mathbb{R}^3)$ is obvious thanks to the energy inequality. A simple interpolation gives $u\in L^{10/3}$. For the second derivatives of weak solutions, a rough estimate $\nabla^2u\in L^{5/4}$ can be obtained by considering $(u\cdot\nabla)u$ as a source term from the standard parabolic regularization theory (see Ladyženskaja, Solonnikov and Ural'ceva [25]). With different ideas, Constantin [12] showed $\nabla^2u\in L^{\frac{4}{3}-\epsilon}$ for any small $\epsilon>0$ in periodic setting, and later Lions [29] improved it up to $\nabla^2u\in$ weak-$L^{\frac{4}{3}}$ (or $L^{\frac{4}{3},\infty}$) by assuming that ∇u_0 is lying in the space of all bounded measures in $\mathbb{R}^3$. They used natural structure of the equation with some interpolation technique. On the other hand, Foiaş, Guillopé and Temam [18] and Duff [15] obtained other kinds of estimates for higher derivatives of weak solutions while Giga and Sawada [19] and Dong and Du [14] covered mild solutions. For asymptotic behavior, we refer to Schonbek and Wiegner [34].

Recently in [41], it has been shown that, for any small $\epsilon>0$, any integer $d\geqslant1$ and any smooth solution u on $(0,T)$, there exist uniform bounds on ∇^du in $L^{\frac{4}{d+1}-\epsilon}_{loc}$, which depend only on the L^2-norm of the initial data once ϵ, d and the domain of integration are fixed. It can be considered as a natural extension of the result of Constantin [12] for higher derivatives. However, the method is very different. In [41], the proof uses the Galilean invariance of the equation and some regularity criterion of [42], which reproves the famous result of [6] by using a parabolic version of the De Giorgi method [13]. Note that this method gives full regularity to the critical Surface Quasi-Geostrophic equation in [7]. The exponent $p=\frac{4}{d+1}$ appears in a non-linear way from the following invariance of the Navier–Stokes scaling $u_\lambda(t,x)=\lambda u(\lambda^2t,\lambda x)$:

$$\|\nabla^du_\lambda\|^p_{L^p}=\lambda^{-1}\|\nabla^du\|^p_{L^p}. \tag{4}$$

In this paper, our main result improves the above result of [41] in the sense of the following three directions. First, we achieve the limiting case weak-$L^{\frac{4}{d+1}}$ (or $L^{\frac{4}{d+1},\infty}$) as Lions [29] did for second derivatives. Second, we make similar bounds for fractional derivatives as well as classical derivatives. Last, we consider not only smooth solutions but also global-time weak solutions. These three improvements will give us that $\nabla^{3-\epsilon}u$, which is almost third derivatives of weak solutions, is locally integrable on $(0,\infty)\times\mathbb{R}^3$.

Our precise result is the following:

Theorem 1.1. *There exist universal constants $C_{d,\alpha}$ which depend only on integer $d\geqslant1$ and real $\alpha\in[0,2)$ with the following two properties* (I) *and* (II):

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