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# Graphs of maps between manifolds in trace spaces and with vanishing mean oscillation

Emilio Acerbi *, Domenico Mucci

*Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, I-43124 Parma, Italy*

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**Abstract**

We give a positive answer to a question raised by Alberti in connection with a recent result by Brezis and Nguyen. We show the existence of currents associated with graphs of maps in trace spaces that have vanishing mean oscillation. The degree of such maps may be written in terms of these currents, of which we give some structure properties. We also deal with relevant examples.


*Keywords:* Vanishing mean oscillation; Currents; Trace spaces

In this paper we give a positive answer to a question raised by Alberti in connection with a recent result by Brezis and Nguyen.

In recent years there has been a growing interest concerning the notion of *degree* for mappings which do not possess the classical regularity properties.

An example of this are the papers [8,9] in which Brezis and Nirenberg investigate the class VMO of functions with *vanishing mean oscillation*, i.e., functions whose mean oscillation on balls (that is, the average of the difference from the integral average) converges to zero with the radius of the ball:

$$\sup_{x_0} \fint_{B_r(x_0)} \left|u-(u)_{B_r(x_0)}\right| \to 0,$$

see Section 1 for more details.

Very recently, Brezis and Nguyen [7] dealt with *continuity properties* for the degree in the same framework. In particular, they show the continuity of the degree of maps from the $n$-dimensional unit sphere $\mathbb{S}^n$ to itself, under the joint convergence in the BMO and $W^{1-1/n,n}$ norms, where BMO is the spaces of functions whose mean oscillation on balls is just bounded.

* Corresponding author.
*E-mail addresses:* emilio.acerbi@unipr.it (E. Acerbi), domenico.mucci@unipr.it (D. Mucci).

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We recall that the degree of a smooth map $u : \mathbb{S}^n \to \mathbb{S}^n$ is defined by

$$\deg(u) := \frac{1}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} \det \nabla u(x)\, d\sigma,$$

where $\det \nabla u = \det(\nabla u, u)$, viewing $(\nabla u, u)$ as a square matrix of order $n + 1$. We thus have

$$\det \nabla u(x) = u^{\#}\omega_n(x), \quad \omega_n := \sum_{j=1}^{n+1} (-1)^{j-1} y^j\, \widehat{dy^j},$$

where $\widehat{dy^j} := dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^{n+1}$. Therefore, by the *area formula* we get

$$\left|\mathbb{S}^n\right| \cdot \deg(u) = \int_{\mathbb{S}^n} u^{\#}\omega_n = \int_{\mathcal{G}_u} \omega_n,$$

where $\mathcal{G}_u$ is the graph of $u$. Therefore, in the smooth case the degree can be written in terms of the current associated to the graph of $u$.

Alberti analyzed the paper [7] in his report on Mathematical Reviews [2], and he raised some interesting questions which are the starting point of our work. In particular, he discussed the possible existence of a *current* $T_u$ associated with the graph of a VMO-map $u$ from $\mathbb{S}^n$ to $\mathbb{S}^n$ belonging also to the space $W^{1-1/n,n}(\mathbb{S}^n, \mathbb{R}^{n+1})$ of *traces* of Sobolev functions $W^{1,n}(B^{n+1}, \mathbb{R}^{n+1})$.

Using the classical extension $U$ introduced by Gagliardo [11] and used e.g. by Bethuel and Demengel [6] for Sobolev classes between manifolds, we give a positive answer to this question by proving, see Theorem 3.1 below (where, as in the rest of the paper, we deal with the general case of maps between Riemannian manifolds $\mathcal{X}$ and $\mathcal{Y}$), that the current

$$T_u := (-1)^{n-1}(\partial G_U) \llcorner \left(\mathbb{S}^n \times \mathbb{S}^n\right)$$

is well-defined. Notice that the degree in the sense of Brezis and Nguyen [7] agrees with the action of our current $T_u$ on the form $\omega_n$, as in the smooth case. An important feature is that the average $(u)_{B_r(x_0)}$ has small distance from the target sphere $\mathbb{S}^n$, independently of the centers of the balls, provided that the radius is small; a similar argument was used also in the approximation theorem by Schoen and Uhlenbeck [16].

Our approach is therefore based on the extension of $u : \mathbb{S}^n_x \to \mathbb{S}^n_y$ to the unit ball $B^{n+1}$ by suitable averages in the spirit of Gagliardo [11]; the VMO condition allows then to modify the extension in a neighbourhood of $\mathbb{S}^n_x$ in order to preserve the constraint on the image.

The current $T_u$ associated to the graph of such a map $u$ is an *integral flat chain*, but in general it may have infinite mass, see Example 3.9. The current $T_u$ acts on $n$-forms defined in $\mathbb{S}^n_x \times \mathbb{S}^n_y$, and it decomposes as $T_u = T_{u(0)} + T_{u(1)} + \cdots + T_{u(n)}$, where $T_{u(j)}$ acts on forms with $j$ vertical differentials $dy$. It turns out that $u$ is a function of *bounded variation* if and only if the first non-trivial component $T_{u(1)}$ has finite mass, see Proposition 3.4. In addition, $T_u$ is a *Cartesian current* in the sense of Giaquinta, Modica and Souček [12] provided it has finite mass, see Proposition 4.1. Moreover, something on the higher order components may be said in the case of Sobolev $W^{1,q}$-maps, see Remark 3.8.

We also show that if a sequence of maps in $W^{1-1/n,n} \cap \mathrm{VMO}$ converges strongly in BMO and in $W^{1-1/n,n}$, then the corresponding graphs weakly converge (in the sense of currents) to the graph of the limit map, see Theorem 5.2. This extends the continuity property of the degree by Brezis and Nguyen [7].

We now mention some open questions raised in this context. From the work of Giaquinta, Modica and Souček [12], it turns out that in order to deal with currents carried by graphs of non-smooth maps $u$, a fundamental property is the *approximate differentiability a.e.* A function of bounded variation is approximately differentiable a.e., compare [5], but we do not know if the same holds true for functions in trace spaces $W^{1-1/p,p}$.

More precisely, in dimension two there is a function $f$ in $C^{1,\alpha}$ for each $0 < \alpha < 1$ that does not satisfy the so-called *weak Sard property*, see [3]. Correspondingly, the function $b = \nabla^{\perp} f$ belongs to the fractional Sobolev classes $W^{s,p}$ for each $p > 1$ and $0 < s < 1$. Therefore, such a function $b$ does not belong to the class $t^{1,1}$ (functions with first order Taylor expansion in $L^1$-sense), see [4]. If it were the case, in fact, the corresponding function $f$ had to satisfy the $C^2$-Lusin property and, definitely, the weak Sard property. Notice that the existence of a first order Taylor expansion