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Existence of immersed spheres minimizing curvature functionals in non-compact 3-manifolds

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Abstract

We study curvature functionals for immersed 2-spheres in non-compact, three-dimensional Riemannian manifold (M,h) without boundary. First, under the assumption that (M,h) is the euclidean 3-space endowed with a semi-perturbed metric with perturbation small in C^1 norm and of compact support, we prove that if there is some point $\bar{x}\in M$ with scalar curvature $R^M(\bar{x})>0$ then there exists a smooth embedding $f:\mathbb{S}^2\hookrightarrow M$ minimizing the Willmore functional $\frac{1}{4}\int|H|^2$, where H is the mean curvature. Second, assuming that (M,h) is of bounded geometry (i.e. bounded sectional curvature and strictly positive injectivity radius) and asymptotically euclidean or hyperbolic we prove that if there is some point $\bar{x}\in M$ with scalar curvature $R^M(\bar{x})>6$ then there exists a smooth immersion $f:\mathbb{S}^2\hookrightarrow M$ minimizing the functional $\int(\frac{1}{2}|A|^2+1)$, where A is the second fundamental form. Finally, adding the bound $K^M\leqslant 2$ to the last assumptions, we obtain a smooth minimizer $f:\mathbb{S}^2\hookrightarrow M$ for the functional $\int(\frac{1}{4}|H|^2+1)$. The assumptions of the last two theorems are satisfied in a large class of 3-manifolds arising as spacelike timeslices solutions of the Einstein vacuum equation in case of null or negative cosmological constant.

MSC: 53C21; 53C42; 58E99; 35J60

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1. Introduction

The present work follows the paper [3] by Kuwert and the authors about the minimization of curvature functionals in Riemannian 3-manifolds under global conditions on the curvature of the ambient space. The aforementioned work is focalized in the case the ambient 3-manifold is compact and develop existence and regularity theory taking inspiration from [16]. The present paper instead is concerned about the non-compact situation and relies on the regularity theory

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developed there. Let us point out that the study of curvature functionals, in particular of the Willmore functional, in the euclidean flat space is a topic of great interest in the contemporary research (see for instance the papers of Li and Yau [8], Kuwert and Schätzle [4], Rivière [13], Simon [16], etc.); the previous [3] and the present work are an attempt to open the almost unexplored field of the corresponding problems in non-constantly curved Riemannian 3-manifolds under global geometric conditions.

Here we consider essentially two problems: first we minimize the Willmore functional among immersed spheres in $\mathbb{R}^3$ endowed with a semi-perturbed metric; second we minimize related curvature functionals in non-compact Riemannian 3-manifolds under global and asymptotic conditions on the metric. As we will remark later in the Introduction the assumptions will include a large class of manifolds naturally arising in General Relativity. Let us start discussing the first problem.

Let $h = h_{\mu\nu}$ be a symmetric bilinear form in $\mathbb{R}^3$ with compact support. Denote by

$$\|h\|_{C^0} := \sup_{x\in\mathbb{R}^3} \sup_{u,v\in S^2} \big|h(x)(u,v)\big|, \qquad \|Dh\|_{C^0} := \sup_{x\in\mathbb{R}^3} \sup_{u,v,w\in S^2} \big|D_w\big(h(x)(u,v)\big)\big|,$$

where D_w is just the directional derivative, and let $\|h\|_{C^1} = \|h\|_{C^0} + \|Dh\|_{C^0}$.

Consider $\mathbb{R}^3$ equipped with the perturbed metric $\delta + h$, where $\delta = \delta_{\mu\nu}$ is the standard euclidean metric. For any immersed closed surface $f : \Sigma \hookrightarrow \mathbb{R}^3$ with induced metric $g = f^*(\delta + h)$, we consider the Willmore functional

$$W(f) = \frac{1}{4}\int_\Sigma |H|^2\, d\mu_g, \tag{1}$$

where H is the mean curvature vector.

The first problem we study is the minimization of $W(f)$ in the class of immersed spheres in the Riemannian manifold $(\mathbb{R}^3, \delta + h)$ and prove the following existence result.

Theorem 1.1. *Assume $\|h\|_{C^0} \leqslant \eta$ and $\|Dh\|_{C^0} \leqslant \theta$, and that* spt $h \subset B^e_{r_0}(x_0)$ *where $B^e_{r_0}(x_0)$ is the ball in euclidean metric of center $x_0 \in \mathbb{R}^3$ and radius $r_0 > 0$. On the class $[\mathbb{S}^2, (\mathbb{R}^3, \delta + h)]$ of smooth immersions $f : \mathbb{S}^2 \to (\mathbb{R}^3, \delta + h)$, consider the Willmore functional*

$$W : \big[\mathbb{S}^2, \big(\mathbb{R}^3, \delta + h\big)\big] \to \mathbb{R}, \qquad W(f) = \frac{1}{4}\int_\Sigma |H|^2\, d\mu_g.$$

Assume that the scalar curvature R_h of $(\mathbb{R}^3, \delta + h)$ is strictly positive in some point $\bar{x} \in \mathbb{R}^3$, namely $R_h(\bar{x}) > 0$. Then for η and $r_0\theta$ sufficiently small there exists a minimizer f in $[\mathbb{S}^2, (\mathbb{R}^3, \delta + h)]$ for W, which is actually an embedding.

In asymptotically flat 3-manifolds, spheres which are critical points of related curvature functionals have been constructed recently by the first author [10,11]; Lamm, Metzger and Schulze [6], see also [5], studied instead the existence of spheres which are critical points of curvature functionals under constraints. They obtain the solutions as perturbations of round spheres using implicit function type arguments.

Among the aforementioned papers, the most related to the present work is [10]; the main difference here (beside the fact that the proofs are completely different, in the former the author used techniques of nonlinear analysis, here we use techniques of geometric measure theory) is that in the former the perturbed metric was C^∞ infinitesimally close to the euclidean metric, then with infinitesimal curvature. Here instead $\delta + h$ is assumed to be close to the euclidean metric δ just in C^0 norm; indeed, in order to have $r_0\theta$ small, $\|Dh\|_{C^0}$ can be large if the support of h is contained in a small ball. Moreover no restrictions are imposed on the derivatives of h of order higher than one, so the Riemann curvature tensor of $(\mathbb{R}^3, \delta + h)$ can be arbitrarily large. For instance, if $h_{\mu\nu}(x) = h_0(x)\delta_{\mu\nu}$ for a certain function $h_0 \in C^\infty_c(\mathbb{R}^3)$, then the perturbed metric $\delta_{\mu\nu} + h_{\mu\nu} = (1 + h_0)\delta_{\mu\nu}$ is conformal to the euclidean metric and a direct computation shows that $R_h = 2\frac{\triangle h_0}{(1+h_0)^2} - \frac{5}{2}\frac{|dh_0|^2}{(1+h_0)^3}$; therefore taking h_0 with small C^1 norm but with large laplacian gives a metric with arbitrarily large curvature which fits in the assumptions of Theorem 1.1 (notice that this example is not trivial since the Willmore functional is invariant under conformal *transformations* of $\mathbb{R}^3$ but *not* under *conformal changes of metric*).