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Counterexample to regularity in average-distance problem

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Abstract

The average-distance problem is to find the best way to approximate (or represent) a given measure μ on $\mathbb{R}^d$ by a one-dimensional object. In the penalized form the problem can be stated as follows: given a finite, compactly supported, positive Borel measure μ, minimize

$$E(\Sigma) = \int_{\mathbb{R}^d} d(x,\Sigma)\,d\mu(x) + \lambda\mathcal{H}^1(\Sigma)$$

among connected closed sets, Σ, where $\lambda > 0$, $d(x,\Sigma)$ is the distance from x to the set Σ, and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. Here we provide, for any $d \geqslant 2$, an example of a measure μ with smooth density, and convex, compact support, such that the global minimizer of the functional is a rectifiable curve which is not C^1. We also provide a similar example for the constrained form of the average-distance problem.

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Keywords: Average-distance problem; Nonlocal variational problem; Regularity

1. Introduction

Given a positive, compactly supported, Borel measure μ on $\mathbb{R}^d$, $d \geqslant 2$, $\lambda > 0$, and Σ a nonempty subset of $\mathbb{R}^d$ consider

$$E(\Sigma) = \int_{\mathbb{R}^d} d(x,\Sigma)\,d\mu(x) + \lambda\mathcal{H}^1(\Sigma). \tag{1}$$

The average-distance problem is to minimize the functional over $\mathcal{A} = \{\Sigma \subset \mathbb{R}^d\colon\ \Sigma -$ connected and compact$\}$.

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The problem was introduced by Buttazzo, Oudet, and Stepanov [1] and Buttazzo and Stepanov [2]. They studied the problem in the constrained form, where instead of $\mathcal{H}^1$ penalization one minimizes

$$F(\Sigma) = \int_{\mathbb{R}^d} d(x, \Sigma)\, d\mu(x) \quad \text{over } \mathcal{A}_1 := \{\Sigma \in \mathcal{A}\colon\ \mathcal{H}^1(\Sigma) \leqslant \ell\}. \tag{2}$$

Over the past few years there has been a significant progress on understanding of the functional; some of which we outline below. An excellent overview article has recently been written by Lemenant [3].

The problem has wide ranging applications. When interpreted as a simplified description of designing the optimal public transportation network then μ represents the distribution of passengers, and Σ is the network. The desire is to design the network that minimizes the total distance of passengers to the network. Another related problem which can be reduced to the average-distance problem, studied in [1], is when we think of passengers as workers that need to get to their workplace. Then two measures are initially given, the distribution of where workers reside and where they work. Again the goal is to find the optimal network that minimizes the total transportation cost (traveling along the network is for free).

A related interpretation is that of finding the optimal irrigation network (the *irrigation problem*).

Another interpretation, whose application in a related setting is presently investigated by Laurent and the author, is to find a good one-dimensional representation to a data cloud. Here μ represents the distribution of data points. One wishes to approximate the cloud by a one-dimensional object. The first term in (1) then charges the errors in the approximation, while the second one penalizes the complexity of the representation.

The existence of minimizers of E follows from the theorems of Blaschke and Gołąb [2]. In this paper we investigate their regularity. It was shown in [2] that, at least for $d = 2$, the minimizer is topologically a tree made of finitely many simple rectifiable curves which meet at triple junctions (no more that three branches can meet at one point). The authors also show that the minimizer is Ahlfors regular (which was extended to higher dimensions by Paolini and Stepanov [4]), but further regularity of branches remained open. Recently Tilli [5] showed that every compact simple $C^{1,1}$ curve is a minimizer of the average-distance problem (in the constrained form (27)) where μ is the characteristic function of a small tubular neighborhood of the curve. This suggests that $C^{1,1}$ is the best regularity for minimizers one can expect (even if μ were smooth). Further criteria for regularity were established by Lemenant [6].

Due to the presence of the $\mathcal{H}^1$ term one might expect that, if μ is a measure with smooth density, Σ is at least C^1. A recent paper by Buttazzo, Mainini, and Stepanov [7] suggests that this may not be the case, and exhibits a measure μ which is a characteristic function of a set in $\mathbb{R}^2$, for which there exists a stationary point of E which has a corner. Furthermore the results on the blow-up of the problem by Santambrogio and Tilli [8] support the possibility of corners. Here we prove that minimizers which are not C^1 are indeed possible. That is for any $d \geqslant 2$ provide an example of a measure μ with smooth density for which we prove that the minimizer is a curve which has a corner, and is thus not C^1. One of the difficulties in dealing with global energy minimizers is that the functional is not convex. To be able to treat them we introduce constructions and an approximation technique that may be of independent interest.

Our approach is based on approximating the measure μ of our interest by particle measures μ_n (i.e. the ones that have only atoms). For particle measures μ_n the average-distance problem (1) has a discrete formulation that can be carefully analyzed. In particular the minimizers are trees with piecewise linear branches. Our starting point is the construction of a particle measure with three particles, $\bar{\mu}$, for which we can show that the minimizer is a wedge (curve with exactly two line segments), see Fig. 1. We then show that if $\bar{\mu}$ is smoothed out a bit then the minimizer will still have a corner (even if we also add a smooth background measure of small total mass, q, that makes the support of the perturbed measure convex). We denote the smooth perturbed measure by $\mu_{q,\delta}$ were δ is the smoothing parameter. To show that a minimizer of E for $\mu_{q,\delta}$ has a corner when δ and q are small, we consider discrete approximations $\mu_{q,\delta,n}$ of $\mu_{q,\delta}$. We show that the minimizers $\Sigma_{q,\delta,n}$ of E corresponding to $\mu_{q,\delta,n}$ have a corner whose opening is bounded from above independent of n. We furthermore obtain appropriate estimates on the minimizers which guarantee convergence as $n \to \infty$ to a minimizer $\Sigma_{q,\delta}$ of E corresponding to $\mu_{q,\delta}$ and insure that the corner remains in the limit.

1.1. Outline

In Section 2 we list some of the basic properties of the functional E given in (1), in particular its continuity properties with respect to parameters and scaling with respect to dilation of μ. In Section 3 we consider the energy (1)