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New results on Γ-limits of integral functionals

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Abstract

For $\psi \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $g \in W^{-1,p}(\Omega;\mathbb{R}^d)$, $1 < p < +\infty$, we consider a sequence of integral functionals $F_k^{\psi,g}: W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n}) \to [0,+\infty]$ of the form

$$F_k^{\psi,g}(u,v) = \begin{cases} \int_\Omega f_k(x,\nabla u, v)\,dx & \text{if } u-\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ and } \operatorname{div} v = g, \\ +\infty & \text{otherwise,} \end{cases}$$

where the integrands f_k satisfy growth conditions of order p, uniformly in k. We prove a Γ-compactness result for $F_k^{\psi,g}$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n})$ and we show that under suitable assumptions the integrand of the Γ-limit is continuously differentiable. We also provide a result concerning the convergence of momenta for minimizers of $F_k^{\psi,g}$.

Résumé

Pour tout $\psi \in W^{1,p}(\Omega;\mathbb{R}^m)$ et $g \in W^{-1,p}(\Omega;\mathbb{R}^d)$, $1 < p < +\infty$, nous considérons une suite de fonctionnelles intégrales $F_k^{\psi,g}: W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n}) \to [0,+\infty]$ définies par

$$F_k^{\psi,g}(u,v) = \begin{cases} \int_\Omega f_k(x,\nabla u, v)\,dx & \text{si } u-\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ et } \operatorname{div} v = g, \\ +\infty & \text{sinon,} \end{cases}$$

où les intégrandes f_k satisfont des conditions de croissance d'ordre p, uniformément en k. Nous démontrons un résultat de Γ-compacité pour $F_k^{\psi,g}$ par rapport à la topologie faible sur $W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n})$ et nous prouvons que sous des conditions appropriées, l'intégrande de la Γ-limite est continûment différentiable. Nous montrons également un résultat de convergence des moments pour les minima de $F_k^{\psi,g}$.

MSC: 35D99; 35E99; 49J45

Keywords: Γ-convergence; Integral functionals; Localization method; (curl, div)-quasiconvexity; Convergence of minimizers; Convergence of momenta

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1. Introduction

In this paper we consider sequences of integral functionals $F_k : W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n}) \to [0,+\infty)$ of the form

$$F_k(u,v) = \int_\Omega f_k(x,\nabla u, v)\,dx, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$ and the integrands f_k satisfy suitable coerciveness and growth conditions of order $p \in (1,\infty)$, uniformly in k (see (2.1) below). Specifically we are interested in the asymptotic behavior of the solutions to the following minimization problems

$$\min\bigl\{F_k(u,v):\ u-\psi \in W_0^{1,p}\bigl(\Omega;\mathbb{R}^m\bigr),\ v \in L^p\bigl(\Omega;\mathbb{R}^{d\times n}\bigr),\ \operatorname{div} v = g\bigr\}, \tag{1.2}$$

where $\psi \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $g \in W^{-1,p}(\Omega;\mathbb{R}^d)$ are given. The relevance of this setting where the functionals are defined on pairs (u,v) satisfying the differential constraint $(\operatorname{curl} u, \operatorname{div} v) = (0,g)$ lies in the fact that in many applications (see e.g. the case of electromagnetism) PDEs constraints of this type arise naturally.

To take into account the boundary and divergence constraints on u and v, respectively, we introduce the functionals

$$F_k^{\psi,g}(u,v) = \begin{cases} F_k(u,v) & \text{if } u-\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ and } \operatorname{div} v = g,\\ +\infty & \text{otherwise.}\end{cases} \tag{1.3}$$

The main result of the present paper is as follows: if f_k is a sequence of functions satisfying (2.1) and F_k are as in (1.1), there exist a subsequence of f_k, not relabeled, and a function f such that the sequence $F_k^{\psi,g}$ Γ-converges to the corresponding functional $F^{\psi,g}$, with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n})$. Moreover, the integrand f does not depend on ψ and g (see Theorem 2.1 and Theorem 3.3). The case of functionals independent of u, with the constraint $\operatorname{div} v = 0$, has been studied in [2], while similar problems in the framework of $\mathcal{A}$-quasiconvexity have been studied in [9]. However, it does not seem that the techniques used in these papers can lead directly, in our case, to a limit integrand f independent of g.

We prove our main result in a nonconstructive way, following the so-called localization method of Γ-convergence. To this end, for every open set $U \subseteq \Omega$ we consider the functionals

$$F_k(u,v,U) = \int_U f_k(x,\nabla u, v)\,dx. \tag{1.4}$$

Notice that at this first stage both the boundary condition $u = \psi$ and the constraint $\operatorname{div} v = g$ are omitted. In order to add the divergence constraint in the final step of the proof, it is convenient to introduce the following distance

$$d\bigl((u_1,v_1),(u_2,v_2)\bigr) := \|u_1-u_2\|_{L^p(\Omega;\mathbb{R}^m)} + \|v_1-v_2\|_{W^{-1,p}(\Omega;\mathbb{R}^{d\times n})} + \bigl\|\operatorname{div}(v_1-v_2)\bigr\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} \tag{1.5}$$

for which $(W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n}), d)$ is separable. Then, thanks to the general theory of Γ-convergence in separable metric spaces, in Section 2 we prove that there exist a subsequence of f_k, not relabeled, and a function f such that for every open set $U \subseteq \Omega$ the functionals $F_k(\cdot,\cdot,U)$ $\Gamma(d)$-converge to the functional $F(\cdot,\cdot,U)$ corresponding to f (see Theorem 2.3). In particular this gives the $\Gamma(d)$-convergence for $U = \Omega$ (see Theorem 2.1). We also prove that under suitable assumptions on f_k, for a.e. $x \in \Omega$ the integrand $f(x,\cdot,\cdot)$ of the $\Gamma(d)$-limit F is continuously differentiable (see Theorem 2.8).

By virtue of the above results, in Section 3 we deduce that the functionals $F_k^{\psi,g}$ Γ-converge to $F^{\psi,g}$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n})$. By general properties of Γ-convergence this gives the convergence of minima and minimizers.

In Section 4 we also prove a result about the convergence of momenta for minimizers. Specifically, in Corollary 4.6 we show that, if (u_k,v_k) is a minimizer of $F_k^{\psi,g}$, then there exist a subsequence of (u_k,v_k), not relabeled, and a minimum point (u,v) of $F^{\psi,g}$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, $v_k \rightharpoonup v$ weakly in $L^p(\Omega;\mathbb{R}^{d\times n})$ (convergence of minimizers), and