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Spectral optimization problems with internal constraint

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Abstract

We consider spectral optimization problems with internal inclusion constraints, of the form

$$\min\{\lambda_k(\Omega):\ D\subset\Omega\subset\mathbb{R}^d,\ |\Omega|=m\},$$

where the set D is fixed, possibly unbounded, and λ_k is the k-th eigenvalue of the Dirichlet Laplacian on Ω. We analyze the existence of a solution and its qualitative properties, and rise some open questions.

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1. Introduction

A spectral optimization problem is a minimization problem of the form

$$\min\{J(\Omega):\ \Omega\in\mathcal{A}\} \tag{1.1}$$

where J is a cost functional depending on the spectrum of an elliptic operator defined on the (quasi) open set Ω and $\mathcal{A}$ is a class of admissible domains. A wide literature on the subject is available, dealing with existence, regularity, necessary conditions of optimality, relaxation, explicit solutions and numerical computations of the optimal shapes. We quote for instance the books [7,18,19] and the articles [2,9,17], where the reader may find a complete list of references on the field.

The simplest situation for the existence of a solution of problem (1.1) occurs when the class of admissible domains $\mathcal{A}$ satisfies an *external* inclusion constraint, i.e. consists on quasi-open sets which are supposed *a priori* contained in a given *bounded* open set D of the Euclidean space $\mathbb{R}^d$,

$$\mathcal{A}=\{\Omega:\ \Omega\subset D,\ \Omega \text{ quasi-open}\}.$$

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In this case a general existence result, due to Buttazzo and Dal Maso (see [11]), states that problem (1.1), with the additional constraint $|\Omega| \leqslant m$ on the Lebesgue measure of the competing domains, admits a solution provided the cost functional J satisfies the following conditions:

(i) J is lower semicontinuous for the γ-convergence, suitably defined;
(ii) J is monotone decreasing for the set inclusion.

When the surrounding box D is unbounded the existence result above is no longer true, as some simple examples show. In the case $D = \mathbb{R}^d$ a quite different approach to the proof of the existence of optimal domains has been considered by Bucur in [5,6], using a refined argument related to the Lions concentration-compactness principle (see [22]), and by Mazzoleni and Pratelli in [21] using a more direct approach. However, the latter approach only works in the case $D = \mathbb{R}^d$, while the concentration-compactness approach seems more flexible for our purposes.

In this paper we consider problem (1.1) where the admissible class $\mathcal{A}$ is defined through an *internal* constraint:

$$\mathcal{A} = \{\Omega\colon\ D \subset \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open},\ |\Omega| \leqslant m\}, \tag{1.2}$$

where D is a fixed quasi-open set of finite measure, possibly unbounded. We consider mainly the cases $J(\Omega) = \lambda_k(\Omega)$; the case of general monotone decreasing functionals is at present still open (see Section 6).

In spite of its simplicity, even for cost functionals like $J(\Omega) = \lambda_1(\Omega)$, the existence proof is rather involved, and several interesting questions arise. For this functional, together with the existence of a solution, we prove some global properties for the optimal set: it has to lie at a finite distance from D (in particular the optimal set is bounded, provided D is bounded), it has finite perimeter outside $\overline{D}$, it is an open set as soon as its measure is strictly greater than the measure of (the quasi-connected) D. Local regularity properties, outside $\overline{D}$ are not discussed here, being similar to the bounding box situation, and we refer the reader for instance to [4]. We discuss as well the existence question for $J(\Omega) = \lambda_k(\Omega)$. We refer the reader to [6] and to [21] for the analysis of these functionals in the absence of any inclusion constraint in $\mathbb{R}^d$.

It is convenient for our purposes to consider also the problem

$$\min\{\lambda_k(\Omega) + \Lambda|\Omega|\colon\ D \subset \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open}\}, \tag{1.3}$$

where the measure constraint $|\Omega| \leqslant m$ is replaced by the Lagrange multiplier penalization $\Lambda|\Omega|$. The relations between the constrained problem

$$\min\{\lambda_k(\Omega)\colon\ D \subset \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open}, |\Omega| \leqslant m\}, \tag{1.4}$$

and the penalized version (1.3) have been analyzed for $k = 1$ in [4], while for general k only a partial result is available (see Lemma 5.10), which is enough for our purposes.

The existence of an optimal domain for problem (1.4), as well as for its penalized version (1.3), is proved in Theorem 4.7.

2. Notations and preliminaries

We introduce here the main tools we use; further details can be found for instance in [7,9].

In the sequel, we will work in the Euclidean space $\mathbb{R}^d$ with $d \geqslant 2$. Given a subset $E \subset \mathbb{R}^d$ we define the capacity of E by

$$\mathrm{cap}(E) = \inf\left\{\int |\nabla u|^2\,dx + \int u^2\,dx\colon\ u \in \mathcal{U}_E\right\},$$

where $\mathcal{U}_E$ is the set of all functions u of the Sobolev space $H^1(\mathbb{R}^d)$ such that $u \geqslant 1$ almost everywhere in a neighborhood of E. If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subset E$ with $\mathrm{cap}(Z) = 0$, we say that $P(x)$ holds *quasi-everywhere* (shortly *q.e.*) on E, whereas the expression *almost everywhere* (shortly *a.e.*) refers, as usual, to the Lebesgue measure, that we often denote by $|\cdot|$.

A subset Ω of $\mathbb{R}^d$ is said to be *quasi-open* if for every $\varepsilon > 0$ there exists an open subset Ω_ε of $\mathbb{R}^d$, with $\Omega \subset \Omega_\varepsilon$, such that $\mathrm{cap}(\Omega_\varepsilon \setminus \Omega) < \varepsilon$. Similarly, a function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *quasi-continuous* (resp. *quasi-lower*