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Pulsating fronts for nonlocal dispersion and KPP nonlinearity

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Abstract

In this paper we are interested in propagation phenomena for nonlocal reaction–diffusion equations of the type:

$$\frac{\partial u}{\partial t} = J * u - u + f(x,u) \quad t \in \mathbb{R},\ x \in \mathbb{R}^N,$$

where J is a probability density and f is a KPP nonlinearity periodic in the x variables. Under suitable assumptions we establish the existence of pulsating fronts describing the invasion of the 0 state by a heterogeneous state. We also give a variational characterization of the minimal speed of such pulsating fronts and exponential bounds on the asymptotic behavior of the solution.

MSC: 45C05; 45G10; 45M15; 45M20; 92D25

Keywords: Periodic front; Nonlocal dispersal; KPP nonlinearity

1. Introduction

In this paper we are interested in propagation phenomena for nonlocal reaction–diffusion equations of the type:

$$\frac{\partial u}{\partial t} = J * u - u + f(x,u) \quad t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{1.1}$$

where J is a probability density and f is a nonlinearity which is KPP in u and periodic in the x variables, that is,

$$f(x,u) = f(x+k,u) \quad \forall x \in \mathbb{R}^N,\ k \in \mathbb{Z}^N,\ u \in \mathbb{R}.$$

More precisely, we are interested in the existence/nonexistence and the characterization of front type solutions called pulsating fronts. A pulsating front connecting 2 stationary periodic solutions p_0, p_1 of (1.1) is an entire solution that has the form $u(x,t) := \psi(x \cdot e + ct, x)$ where e is a unit vector in $\mathbb{R}^N$, $c \in \mathbb{R}$, and $\psi(s,x)$ is periodic in the x variable, and such that

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$$\lim_{s\to-\infty} \psi(s,x) = p_0(x) \quad \text{uniformly in } x,$$
$$\lim_{s\to+\infty} \psi(s,x) = p_1(x) \quad \text{uniformly in } x.$$

The real number c is called the effective speed of the pulsating front.

Using an equivalent definition, pulsating fronts were first defined and used by Shigesada, Kawasaki and Teramoto [58,59] in their study of biological invasions in a heterogeneous environment modeled by the following reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \nabla \cdot \big(A(x)\nabla u\big) + f(x,u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.2}$$

where $A(x)$ and $f(x,u)$ are respectively a periodic smooth elliptic matrix and a smooth periodic function. Using heuristics and numerical simulations, in a one-dimensional situation and for the particular nonlinearity $f(x,u) := u(\eta(x) - \mu u)$, Shigesada, Kawasaki and Teramoto were able to recover earlier results on the minimal speed of spreading obtained by probabilistic methods by Gärtner and Freidlin [34,35].

The above definition of pulsating front has been introduced by Xin [62,63] in his study of flame propagation. This definition is a natural extension of the definition of the sheared traveling fronts studied for example in [10,11]. Within this framework, Xin [62,63] has proved existence and uniqueness up to translation of pulsating fronts for Eq. (1.2) with a homogeneous bistable or ignition nonlinearity. Since then, much attention has been drawn to the study of periodic reaction–diffusion equations and the existence and the uniqueness of pulsating front have been proved in various situations, see for example [5,8,9,38–41,47,61–64]. In particular, Berestycki, Hamel and Roques [8,9] have showed that when $f(x,u)$ is of KPP type, then the existence of a unique nontrivial stationary solution $p(x)$ to (1.2) is governed by the sign of the periodic principal eigenvalue of the following spectral problem

$$\nabla \cdot \big(A(x)\nabla \phi\big) + f_u(x,0)\phi + \lambda_p \phi = 0.$$

Furthermore, they have showed that there exists a critical speed c^* so that a pulsating front with speed $c \geqslant c^*$ in the direction e connecting the two equilibria 0 and $p(x)$ exists and no pulsating front with speed $c < c^*$ exists. They also gave a precise characterization of c^* in terms of some periodic principal eigenvalue. Versions of (1.2) with periodicity in time, or more general media are studied in [5–7,48,50–53,55,66]. It is worth noticing that when the matrix A and f are homogeneous, then Eq. (1.2) reduces to a classical reaction–diffusion equation with constant coefficients and the pulsating front (ψ, c) is indeed a traveling front which have been well studied since the pioneering works of Kolmogorov, Petrovsky and Piskunov [44].

Here we are concerned with a nonlocal version of (1.2) where the classical local diffusion operator $\nabla \cdot (A(x)\nabla u)$ is replaced by the integral operator $J * u - u$. The introduction of such type of long range interaction finds its justification in many problems ranging from micro-magnetism [26–28], neural network [31] to ecology [16,19,29,45,49,60]. For example, in some population dynamic models, such long range interaction is used to model the dispersal of individuals through their environment, [32,33,42]. Regarding Eq. (1.1) we quote [1,2,18,20,21,23,25] for the existence and characterization of traveling fronts for this equation with homogeneous nonlinearity and [3,22,24,36,42] for the study of the stationary problem.

In what follows, we assume that $J : \mathbb{R}^N \to \mathbb{R}$ satisfies

$$\begin{cases} J \geqslant 0, \quad \displaystyle\int_{\mathbb{R}^N} J = 1, \quad J(0) > 0, \\ J \text{ is smooth, symmetric with support contained in the unit ball,} \end{cases} \tag{1.3}$$

and that $f : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is $[0,1]^N$-periodic in x and satisfies

$$\begin{cases} f \in C^3\big(\mathbb{R}^N \times [0,\infty)\big), \\ f(\cdot,0) \equiv 0, \\ f(x,u)/u \text{ is decreasing with respect to } u \text{ on } (0,+\infty), \\ \text{there exists } M > 0 \text{ such that } f(x,u) \leqslant 0 \text{ for all } u \geqslant M \text{ and all } x. \end{cases} \tag{1.4}$$