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Ann. I. H. Poincaré – AN 29 (2012) 21–34

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

An isoperimetric inequality for a nonlinear eigenvalue problem

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Received 17 June 2011; received in revised form 23 July 2011; accepted 8 August 2011

Available online 17 August 2011

Abstract

We prove an isoperimetric inequality of the Rayleigh–Faber–Krahn type for a nonlinear generalization of the first *twisted* Dirichlet eigenvalue, defined by

$$\lambda^{p,q}(\Omega)=\inf\left\{\frac{\|\nabla v\|_{L^p(\Omega)}}{\|v\|_{L^q(\Omega)}},\ v\neq 0,\ v\in W_0^{1,p}(\Omega),\ \int_\Omega |v|^{q-2}v\,dx=0\right\}.$$

More precisely, we show that the minimizer among sets of given volume is the union of two equal balls.

Résumé

On montre une inégalité isopérimétrique du type Rayleigh–Faber–Krahn pour une généralisation non-linéaire de la première valeur propre de Dirichlet *torsadée*, définie par

$$\lambda^{p,q}(\Omega)=\inf\left\{\frac{\|\nabla v\|_{L^p(\Omega)}}{\|v\|_{L^q(\Omega)}},\ v\neq 0,\ v\in W_0^{1,p}(\Omega),\ \int_\Omega |v|^{q-2}v\,dx=0\right\}.$$

Plus précisément, on montre que le minimum parmi les ensembles de volume donné est l'union de deux boules égales.

Keywords: Shape optimization; Eigenvalues; Symmetrization; Euler equation; Shape derivative

1. Introduction

In this note we study a generalized version of the so-called twisted Dirichlet eigenvalue problem. More precisely, for Ω an open bounded subset of $\mathbb{R}^N$ we set

$$\lambda^{p,q}(\Omega)=\inf\left\{\frac{\|\nabla v\|_{L^p(\Omega)}}{\|v\|_{L^q(\Omega)}},\ v\neq 0,\ v\in W_0^{1,p}(\Omega),\ \int_\Omega |v|^{q-2}v\,dx=0\right\}. \tag{1.1}$$

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2011.08.001

Among the sets Ω with fixed volume, we are interested in characterizing those which minimize $\lambda^{p,q}(\Omega)$. In other words we look for an isoperimetric inequality of Rayleigh–Faber–Krahn type. This kind of inequality is related to the optimization of the first eigenvalue for the Dirichlet problem associated to nonlinear operators in divergence form and has been widely studied for functionals that do not involve mean constraints. In such cases a rearrangement technique proves that the minimizing set is a ball and several results concerning its stability are also available (see for instance [23,21,15]). When mean type constraints are considered together with the Dirichlet boundary condition in an eigenvalue problem, the optimization problem becomes more difficult, since one is lead to deal with non-local problems. Due to the fact that an eigenfunction for $\lambda^{p,q}(\Omega)$ is forced to change sign inside Ω, and hence has at least two nodal domains, one cannot expect in general to have a radial optimizer.

The adjective *twisted* was introduced by Barbosa and Bérard in [1], in the study of spectral properties of the second variation of a constant mean curvature immersion of a Riemannian manifold. In that framework a Dirichlet eigenvalue problem arose naturally with a vanishing mean constraint. The condition on the mean value comes from the fact that the variations under consideration preserve some balance of volume.

Further results in this direction can be found in the paper of Freitas and Henrot [14], where, dealing with the linear case, the authors solved the shape optimization problem for the first twisted Dirichlet eigenvalue. In particular they considered $\lambda^{2,2}(\Omega)$, and they proved that the only optimal shape is given by a pair of balls of equal measure. The one-dimensional case has also attracted much interest. In [6], Dacorogna, Gangbo and Subía studied the following generalization of the Wirtinger inequality

$$\inf\left\{\frac{\|u'\|_{L^p((-1,1))}}{\|u\|_{L^q((-1,1))}},\ u\in W_0^{1,p}(-1,1)\setminus\{0\},\ \int_{-1}^{1}|u|^{q-2}u\,dx=0\right\} \tag{1.2}$$

for $p,q>1$ proving that the optimizer is an odd function. Moreover they explained the connection between the value of $\lambda^{p,p'}((-1,1))$, where $p'=\frac{p}{p-1}$, and an isoperimetric inequality. Indeed, let $A\subset\mathbb{R}^2$ whose boundary is a simple closed curve $t\in[-1,1]\to(x(t),y(t))$ with $x,y\in W_0^{1,p}((-1,1))$. Let

$$L(\partial A)=\int_{-1}^{1}\left(\left|x'(t)\right|^p+\left|y'(t)\right|^p\right)^{\frac{1}{p}}dt$$

and

$$M(A)=\frac{1}{2}\int_{-1}^{1}\left[y'(t)x(t)-y(t)x'(t)\right]dt.$$

Then $L^2(\partial A)-4\lambda^{p,p'}((-1,1))M(A)\geqslant 0$. The case of equality holds if and only if $A=\{(x,y)\in\mathbb{R}^2\colon\ |x|^{p'}+|y|^{p'}=1\}$, up to a translation and a dilation.

Several other results are available in the one-dimensional case, see for instance [5,4,2,19,11] and the references therein for further details.

Our aim here, as in [14], is to prove that the optimal shape for $\lambda^{p,q}(\Omega)$ is a pair of equal balls. The main result can be stated as follows.

Theorem 1. *Let Ω be an open bounded subset of $\mathbb{R}^N$. Then, for*

$$1<p<\infty\quad\text{and}\quad\begin{cases}1<q<p^*=\dfrac{Np}{N-p}, & \text{if } 1<p<N,\\ 1<q<\infty, & \text{if } p\geqslant N\end{cases} \tag{1.3}$$

we have

$$\lambda^{p,q}(\Omega)\geqslant\lambda^{p,q}(B_1\cup B_2),$$

where B_1 and B_2 are disjoint balls of measure $|\Omega|/2$.