Available online at www.sciencedirect.com
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Ann. I. H. Poincaré – AN 29 (2012) 377–399
ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Existence of solutions to an initial Dirichlet problem of evolutional $p(x)$-Laplace equations ☆

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Received 1 September 2011; received in revised form 20 December 2011; accepted 10 January 2012

Available online 16 January 2012

Abstract

The existence and uniqueness of weak solutions are studied to the initial Dirichlet problem of the equation

$$u_t = \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) + f(x,t,u),$$

with $\inf p(x) > 2$. The problems describe the motion of generalized Newtonian fluids which were studied by some other authors in which the exponent p was required to satisfy a logarithmic Hölder continuity condition. The authors in this paper use a difference scheme to transform the parabolic problem to a sequence of elliptic problems and then obtain the existence of solutions with less constraint to $p(x)$. The uniqueness is also proved.

Keywords: Electrorheological fluids; $p(x)$-Laplace; Degenerate; Parabolic

1. Introduction

Let $\Omega \subset R^N$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Consider the following problem

$$u_t = \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) + f(x,t,u), \quad x \in \Omega,\ 0 < t < T, \tag{1.1}$$

$$u|_{\Gamma_T} = 0, \qquad u|_{t=0} = u_0, \tag{1.2}$$

where $\Gamma_T = \partial\Omega \times [0,T]$ and $p(x)$ is a measurable function.

In the case when p is a constant, there have been many results about the existence, uniqueness and the regularity of the solutions. We refer the readers to the bibliography given in [5,11,12] and the references therein.

A new interesting kind of fluids of prominent technological interest has recently emerged: the so-called electrorheological fluids. This model includes parabolic equations which are nonlinear with respect to the gradient of the thought solution, and with variable exponents of nonlinearity. The typical case is the so-called evolution p-Laplace equation with exponent p as a function of the external electromagnetic field (see [1,2,10] and the references therein).

☆ The project is supported by NSFC (10771085, 10571072) and by the 985 program of Jilin University.
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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2012.01.001

In [13], Zhikov showed that

$$W_0^{1,p(x)}(\Omega) \neq \{v \in W^{1,p(x)}(\Omega) \mid v|_{\partial\Omega} = 0\} = \mathring{W}^{1,p(x)}(\Omega).$$

Hence, the property of the space is different from the case when p is a constant (see Section 2 for the definition of the function spaces).

As we have known, when p is a constant, the non-degenerate problems have classical solutions and hence the weak solutions exist. But to the case of $p(x)$-Laplace type, there is no results to the corresponding non-degenerate problems. These will bring us some new difficulties in studying the weak solutions.

For more general $p(x,t)$-Laplace equation, the authors of [3] established the existence and uniqueness results with the exponent $p(x,t)$ satisfying the so-called logarithmic Hölder continuity condition, i.e.

$$\left|p(x) - p(y)\right| \leqslant \omega\left(|x-y|\right), \quad \forall x, y \in Q_T,\ |x-y| < \frac{1}{2} \tag{1.3}$$

with

$$\overline{\lim_{s\to 0^+}}\, \omega(s) \ln\left(\frac{1}{s}\right) = C < \infty.$$

However if $p(x,t)$ satisfies (1.3), then (see [14])

$$W_0^{1,p(x)}(\Omega) = \mathring{W}^{1,p(x)}(\Omega).$$

Therefore, we can ask whether the logarithmic Hölder continuity to $p(x,t)$ is indispensable for the existence of solutions to the problem.

In the present work, we will study the existence of the solutions to problem (1.1)–(1.2) without the condition (1.3). Unlike [3] , we will, in this paper, adopt a method of difference in time. Note that the author in [9] considered the p-Laplace equation without the term $f(x,t,u)$ by using a similar method. To overcome the difficulties caused by $p(x)$, we will develop some new ideas and new techniques.

The outline of this paper is the following: In Section 2, we introduce some basic Lebesgue and Sobolev spaces and state our main theorems. In Section 3, we give the existence of weak solutions to a difference equation (approximating problem). In Section 4, we will prove the global existence of solutions to the problem (1.1)–(1.2). Section 5 will be devoted to the proof of the local existence and the existence of weak solutions under some weaker conditions to the initial function u_0.

2. Basic spaces and the main results

To study our problems, we need to introduce some new function spaces.

Denote

$$p^+ = \operatorname{ess}\sup_{\overline{\Omega}} p(x), \qquad p^- = \operatorname{ess}\inf_{\overline{\Omega}} p(x).$$

Throughout the paper we assume that

$$2 < p^- \leqslant p(x) \leqslant p^+ < \infty, \quad \forall x \in \Omega. \tag{2.1}$$

Set

$$L^{p(x)}(\Omega) = \left\{u \;\middle|\; u \text{ is a measurable real-valued function, } \int_\Omega \left|u(x)\right|^{p(x)} dx < \infty\right\},$$

$$|u|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 \;\middle|\; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\},$$

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$

$$|u|_{W^{1,p(x)}} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We use $W_0^{1,p(x)}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}$.

In the following, we state some of the properties of the function spaces introduced as above (see [6] and [7]).