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The symplectic structure of curves in three dimensional spaces of constant curvature and the equations of mathematical physics

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Abstract

The paper defines a symplectic form on an infinite dimensional Fréchet manifold of framed curves over the three dimensional space forms. The curves over which the symplectic form is defined are called horizontal-Darboux curves. It is then shown that the projection on the Lie algebra of the Hamiltonian vector field associated with the functional $f = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$ satisfies Heisenberg's magnetic equation (HME), $\frac{\partial \Lambda}{\partial t}(s,t) = \frac{1}{i}[\Lambda(s), \frac{\partial^2 \Lambda}{\partial s^2}(s,t)]$ in the space of Hermitian matrices for the hyperbolic and the Euclidean case, and $\frac{\partial \Lambda}{\partial t}(s,t) = [\Lambda(s), \frac{\partial^2 \Lambda}{\partial s^2}(s,t)]$ in the space of skew-Hermitian matrices for the spherical case. It is then shown that the horizontal-Darboux curves are parametrized by curves in SU_2, which along the solutions of (HME) satisfy Schroedinger's non-linear equation (NSL)

$$-i\frac{\partial \psi}{\partial t}(t,s) = \frac{\partial^2 \psi}{\partial s^2}(t,s) + \frac{1}{2}\big(|\psi(t,s)|^2 + c\big)\psi(t,s)$$

It is also shown that the critical points of $\frac{1}{2}\int_0^L \kappa^2(s)\,ds$, known as the elastic curves, correspond to the soliton solutions of (NSL). Finally the paper shows that the modifed Korteweg–de Vries equation or the curve shortening equation are Hamiltonian equations generated by $f_1 = \int_0^L \kappa^2(s)\tau(s)\,ds$ and $f_2 = \int_0^L \tau(s)\,ds$ and that $f_0 = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$, f_1 and f_2 are all in involution with each other.

Keywords: Lie groups; Lie algebras; Symmetric spaces; Orthonormal frame bundles; Fréchet spaces; Symplectic forms; Hamiltonian vector fields

1. Introduction

This paper defines a symplectic form on an infinite dimensional Fréchet manifold of framed curves of fixed length over a three dimensional simply connected Riemannian manifold of constant curvature. The framed curves are anchored at the initial point and are further constrained by the condition that the tangent vector of the projected curve coincides with the first leg of the orthonormal frame. Such class of curves are called anchored Darboux curves and in particular include the Serret-Frenet framed curves.

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The symplectic form ω is defined on the space of "horizontal" curves of fixed length in the universal covers of the orthonormal frame bundles of the underlying manifolds: $SL_2(C)$ for the hyperboloid $\mathbb{H}^3$ and $SU_2 \times SU_2$ for the sphere S^3. The form ω is left invariant and is induced by the Poisson–Lie bracket on the appropriate Lie algebra. More precisely, the form ω in each of the above cases is defined over the curves whose tangents take values in the Cartan space $\mathfrak{p}$ corresponding to the decomposition

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$$

of the Lie algebra $\mathfrak{g}$ subject to the usual Lie algebraic relations

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}, \qquad [\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}, \qquad [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}.$$

In the case of the hyperboloid $\mathfrak{g}$ is equal to $sl_2(C)$ and the Cartan space $\mathfrak{p}$ is equal to the space of the Hermitian matrices, while in the case of the sphere $\mathfrak{g}$ is equal to $su_2 \times su_2$ and the Cartan space is isomorphic to the space of skew-Hermitian matrices $\mathfrak{h}$. The symplectic forms in each of these two cases are isomorphic to each other as a consequence of the isomorphism between $\mathfrak{p}$ and $\mathfrak{h}$ given by $i\mathfrak{h} = \mathfrak{p}$.

The Euclidean space $\mathbb{E}^3$ is identified with $\mathfrak{p}$ equipped with the metric defined by the trace form, and its framed curves are represented in the semidirect product $\mathfrak{p} \rtimes SU_2$. The Euclidean Darboux curves inherit the hyperbolic symplectic form ω which is isomorphic to the symplectic form used by J. Millson and B. Zombro in [16].

Each group G mentioned above is a principal SU_2-bundle over the underlying symmetric space with a natural connection defined by the left invariant vector fields that take values in the Cartan space $\mathfrak{p}$. The vertical distribution is defined by the left invariant vector fields that take their values in $\mathfrak{k}$. In this setting then, anchored Darboux curves are the solutions in G of a differential equation

$$\frac{dg}{ds}(s) = g(s)\big(E_1 + u_1(s)A_1 + u(s)A_2 + u_3(s)A_3\big) \tag{1}$$

with $g(0) = I$, where E_1 is a fixed unit vector in the Cartan space $\mathfrak{p}$. The matrices A_1, A_2, A_3 denote the skew-Hermitian Pauli matrices, and $u_1(s), u_2(s), u_3(s)$ are arbitrary real valued functions on a fixed interval $[0, L]$. Each anchored Darboux curve defines a horizontal-Darboux curve $h(s) \in G$ that is a solution of the differential equation

$$\frac{dh}{ds}(s) = h(s)\Lambda(s), \qquad \Lambda(s) = R(s)E_1R^{-1}(s) \tag{2}$$

with $R(s)$ the solution curve in SU_2 of the equation

$$\frac{dR}{ds} = R(s)\big(u_1(s)A_1 + u_2(s)A_2 + u_3(s)A_3\big) \tag{3}$$

that satisfies $R(0) = I$. The symplectic form for the hyperbolic Darboux curves is given by

$$\omega_\Lambda(V_1, V_2) = \frac{1}{i}\int_0^L \big\langle \Lambda(s), \big[U_1(s), U_2(s)\big]\big\rangle ds \tag{4}$$

with $U_1(s)$ and $U_2(s)$ Hermitian matrices orthogonal to the tangent vector $\Lambda(s)$, that further satisfy $U_j(0) = 0$ and $\frac{dV_j}{ds}(s) = U_j(s)$ for $j = 1, 2$.

In the spherical case the symplectic form has the same form as in the hyperbolic case, except for the factor $\frac{1}{i}$, which is omitted. The matrices U_j in this case take values in $\mathfrak{k}$ and satisfy

$$\frac{dV_j}{ds}(s) = \big[\Lambda(s), V_j(s)\big] + U_j(s)$$

for $j = 1, 2$.

The second part of the paper is devoted to the Hamiltonian flow associated with the function

$$f\big(g(s)\big) = \frac{1}{2}\int_0^L \left\| \frac{d\Lambda}{ds}(s) \right\|^2 ds = \frac{1}{2}\int_0^L \kappa^2(s)\, ds$$