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Multiple critical points of perturbed symmetric strongly indefinite functionals

Points critiques multiples de perturbations de fonctionnelles symétriques fortement indéfinies

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Abstract

We prove that the elliptic system

$$-\Delta u = |v|^{q-2}v + k(x), \quad x \in \Omega, \tag{1}$$

$$-\Delta v = |u|^{p-2}u + h(x), \quad x \in \Omega, \tag{2}$$

where Ω is a regular bounded domain of $\mathbb{R}^N$, $N \geqslant 3$ and $h, k \in L^2(\Omega)$, admits an unbounded sequence of solutions $(u_k, v_k) \in H_0^1(\Omega) \times H_0^1(\Omega)$, provided $2 < p \leqslant q$ and $\frac{N}{2}(1 - \frac{1}{p} - \frac{1}{q}) < \frac{p-1}{p}$. We also prove a generic multiplicity result for exponents in the open region bounded by the lines $p = 2$, $q = 2$ and the critical hyperbola.

Résumé

Nous démontrons que le système elliptique ((1), (2)) où Ω est un domaine régulier de $\mathbb{R}^N$, $N \geqslant 3$ et $h, k \in L^2(\Omega)$, possède une suite non bornée de solutions $(u_k, v_k) \in H_0^1(\Omega) \times H_0^1(\Omega)$, pour autant que $2 < p \leqslant q$ et $\frac{N}{2}(1 - \frac{1}{p} - \frac{1}{q}) < \frac{p-1}{p}$. Nous démontrons également un résultat générique de mulplicité lorsque les exposants se situent dans l'ouvert délimité par les droites $p = 2$, $q = 2$ et l'hyperbole critique.

MSC: 35J50; 35J55; 58E05

Keywords: Elliptic system; Strongly indefinite functional; Perturbation from symmetry; Lyapunov–Schmidt reduction; Genericity

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1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 3$, and $h, k \in L^2(\Omega)$. We consider an elliptic system of the form

$$\begin{cases} -\Delta u = |v|^{q-2}v + k(x) & \text{in } \Omega, \\ -\Delta v = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $p, q > 2$. Here q stands for the largest exponent appearing in (1.1), that is we assume without loss of generality that $p \leqslant q$. In case $h(x) = k(x)$ and $p = q > 2$, the system reduces to a single equation

$$-\Delta u = |u|^{p-2}u + h(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

This equation can be seen as a (large) perturbation of an equation possessing a natural $\mathbb{Z}_2$-symmetry and thus a large number of solutions are expected. One can indeed obtain infinitely many solutions, provided the growth range of the nonlinearity is suitably restricted. Namely, Bahri and Berestycki [3], Struwe [24], and, with a different approach, Rabinowitz [16,17] proved the existence of infinitely many solutions for problem (1.2) under the restriction

$$\frac{2}{p} + \frac{1}{p-1} > \frac{2N-2}{N}, \tag{1.3}$$

while, later on, Bahri and Lions [4] and Tanaka [26] (see also [14]) showed that it is sufficient to assume

$$p < \frac{2N-2}{N-2}. \tag{1.4}$$

Moreover, assuming the "natural" growth restriction $p < 2N/(N-2)$, Bahri [2] proved that there is an open dense set of functions $h \in H^{-1}(\Omega)$ for which (1.2) admits infinitely many weak solutions. We also mention that the radially symmetric case has been studied by Kajikiya [13] and Struwe [25] while Tehrani [28] dealt with sign-changing nonlinearities. More recent results, including nonhomogeneous boundary conditions and information on the sign of the solutions, can be found in [5,6,21] and their references.

In the past years, a special attention has been devoted to the study of elliptic systems leading to strongly indefinite functionals. In the context of superlinear elliptic systems with perturbed symmetry, we mention the recent papers by Clapp, Ding and Hernández-Linares [7] and by de Figueiredo and Ding [10] which deal with potential systems of the form

$$-\Delta u = \partial_u F(x, u, v) \quad \text{in } \Omega,$$
$$\Delta v = \partial_v F(x, u, v) \quad \text{in } \Omega,$$

for some smooth function $F(x, u, v)$.

For strongly coupled systems such as (1.1), we are merely aware of the works [1,27]. In [1], Angenent and van der Vorst showed, among other results, that the unperturbed system (with $h(x) = 0 = k(x)$) admits an unbounded sequence of solutions under the "natural" restriction (cf. [8,11,12])

$$\frac{N}{2}\left(1 - \frac{1}{p} - \frac{1}{q}\right) < 1, \tag{1.5}$$

while Tarsi [27] proved that the same conclusion holds for the perturbed system (1.1) under the restriction (recall also that $2 < p \leqslant q$)

$$\frac{1}{p} + \frac{1}{q} + \frac{p}{(p-1)q} > \frac{2N-2}{N}. \tag{1.6}$$

We observe that (1.6) implies condition (1.3) (and it reduces to (1.3) in case $p = q$); in particular, p is not allowed to be close to the critical range $(2N-2)/(N-2)$ which appears in (1.4). Observe also that (1.6) implies both p and q to be smaller than the critical Sobolev exponent $2N/(N-2)$.

In the present note we extend the main result in [27] by proving the following.