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A kinetic model for coagulation–fragmentation

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Abstract

The aim of this paper is to show an existence theorem for a kinetic model of coagulation–fragmentation with initial data satisfying the natural physical bounds, and assumptions of finite number of particles and finite L^p-norm. We use the notion of renormalized solutions introduced by DiPerna and Lions (1989) [3], because of the lack of *a priori* estimates. The proof is based on weak-compactness methods in L^1, allowed by L^p-norms propagation.

1. Introduction

Coalescence and fragmentation are general phenomena which appear in dynamics of particles, in various fields (polymers chemistry, raindrops formation, aerosols, ...). We can describe them at different scales, which lead to different mathematical points of view. First, we can study the dynamics at the microscopic level, with a system of N particles which undergo successives mergers/break ups in a random way. We refer to the survey [1] for this stochastic approach. Another way to describe coalescence and fragmentation is to consider the statistical properties of the system, introducing the statistical distribution of particles $f(t,m)$ of mass $m>0$ at time $t \geqslant 0$ and studying its evolution in time. This approach is rather macroscopic. But we can put in an intermediate level, by considering a density f which depends on more variables, like position x or velocity v of particles, and this description is more precise. Here, we start by discussing models with density, from the original (with $f=f(t,m)$) to the kinetic one (with $f=f(t,x,m,v)$), which is the setting of this work.

Depending on the physical context, the mass variable is discrete (polymers formation) or continuous (raindrops formation). It leads to two sorts of mathematical models, with $m \in \mathbb{N}^\star$ or $m \in (0,+\infty)$, but we focus on the continuous case. To understand the relationship between discrete and continuous equations, see [16].

1.1. The original model

The discrete equations of coagulation have been originally derived by Smoluchowski in [21,22], by studying the Brownian motion of colloidal particles. It had been extended to the continuous setting by Müller [20], giving the

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following mathematical model, called the *Smoluchowski's equation of coagulation*:

$$\frac{\partial f}{\partial t}(t,m) = Q_c^+(f,f) - Q_c^-(f,f), \quad (t,m) \in (0,+\infty)^2. \tag{1.1}$$

This equation describes the evolution of the statistical mass distribution in time. At each time $t > 0$, the term $Q_c^+(f,f)$ represents the gain of particles of mass m created by coalescence between smaller ones, by the reaction

$$\{m^\star\} + \{m - m^\star\} \to \{m\}.$$

The term $Q_c^-(f,f)$ is the depletion of particles of mass m because of coagulation with other ones, following the reaction

$$\{m\} + \{m^\star\} \to \{m + m^\star\}.$$

Namely, we have

$$\begin{cases} Q_c^+(f,f)(t,m) = \dfrac{1}{2}\displaystyle\int_0^m A(m^\star, m - m^\star) f(t,m^\star) f(t, m - m^\star)\, dm^\star, \\ Q_c^-(f,f)(t,m) = \displaystyle\int_0^{+\infty} A(m, m^\star) f(t,m) f(t,m^\star)\, dm^\star, \end{cases}$$

where $A(m,m^\star)$ is the coefficient of coagulation between two particles, which governs the frequency of coagulations, according to the mass of clusters. In his original model, Smoluchowski derived the following expression for A:

$$A(m,m^\star) = (m^{1/3} + m^{\star 1/3})(m^{-1/3} + m^{\star -1/3}). \tag{1.2}$$

In many cases, coalescence is not the only mechanism governing the dynamics of particles, and other effects should be taken into account. A classical phenomenon which also occurs is the fragmentation of particles in two (or more) clusters, resulting from an internal dynamic (we do not deal here with fragmentation processes induced by particles collisions). This binary fragmentation is modeled by linear additional reaction terms in Eq. (1.1), namely

$$\begin{cases} Q_f^+(f)(t,m) = \displaystyle\int_m^{+\infty} B(m', m) f(t, m')\, dm', \\ Q_f^-(f)(t,m) = \dfrac{1}{2} f(t,m) B_1(m), \quad \text{where } B_1(m') = \displaystyle\int_0^{m'} B(m', m)\, dm. \end{cases}$$

The function $B(m', m)$ is the fragmentation kernel, it measures the frequency of the break-up of a mass m' in two clusters m and $m' - m$, for $m < m'$. So, at each time t, the term $Q_f^+(f)$ is the gain of particles of mass m, resulting from the following reaction of fragmentation:

$$\{m'\} \to \{m\} + \{m' - m\},$$

whereas $Q_f^-(f)$ stands for the loss of particles of mass m, because of a break-up into two smaller pieces, by the following way:

$$\{m\} \to \{m^\star\} + \{m - m^\star\}, \quad \text{with } m^\star < m.$$

Thus, the continuous *coagulation–fragmentation equation* writes

$$\frac{\partial f}{\partial t}(t,m) = Q_c^+(f,f) - Q_c^-(f,f) + Q_f^+(f) - Q_f^-(f), \quad (t,m) \in (0,+\infty)^2. \tag{1.3}$$

In the 90's, many existence and uniqueness results have been proved about this problem, see for instance [23], [24], or [13] for an approach by the semigroups of operators theory. These results are true under various growth hypotheses