Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 23 (2006) 683–694
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Essential Dynamics for Lorenz maps on the real line and the Lexicographical World ☆

Rafael Labarca [a,*], Carlos Gustavo Moreira [b]

[a] *Departamento de Matematica y CC, Universidad de Santiago de Chile, Casilla 307 Correo 2, Santiago, Chile*
[b] *IMPA, Estrada Dona Castorina 110, CEP 22460-320, Jardim Botanico, Rio de Janeiro, Brasil*

Received 11 March 2005; received in revised form 11 July 2005; accepted 19 September 2005

Available online 19 January 2006

Dedicated to the memory of Professor Jorge Billeke G.

Abstract

In this paper we describe some topological and geometric properties of the set of sequences $LW = \{(a,b) \in \Sigma_0 \times \Sigma_1;\ a \leqslant \sigma^n(a) \leqslant b,\ a \leqslant \sigma^n(b) \leqslant b,\ \forall n \in \mathbb{N}\}$, which essentially represents all the allowed dynamics for piecewise continuous increasing maps with one discontinuity. In particular, we describe the first main bifurcations in *LW* which generate non-trivial dynamics, and we study (fractal) geometric properties of *LW* and of the phase spaces $\Sigma_{a,b}$ associated to it.

Résumé

Dans ce travail nous décrivons quelques proprietés topologiques et géometriques de l'ensemble de suites $LW = \{(a,b) \in \Sigma_0 \times \Sigma_1;\ a \leqslant \sigma^n(a) \leqslant b,\ a \leqslant \sigma^n(b) \leqslant b,\ \forall n \in \mathbb{N}\}$, que répresentent essentiellement toutes les dynamiques permises pour des fonctions continues et croissantes par morceaux avec un point de discontinuité. En particulier, on décrit les premières bifurcations dans *LW* qui produisent des dynamiques non-triviales et nous étudions des proprietés géometriques (fractales) de *LW* et des espaces de phase $\Sigma_{a,b}$ associés.

Keywords: Essential Dynamics; Lexicographical World; Lorenz maps; Hausdorff dimension; Entropy

1. Introduction

In the remarkable work [13], a meteorologist, E.N. Lorenz, showed numerical evidence of the existence of a strange attractor for a quadratic system of ordinary differential equations in three variables. Some time later J. Guckenheimer, [7], produced a work where he introduced symbolic dynamics in order to understand the topological equivalence classes for nearly similar attractors. At that time R.F. Williams, [26], introduced a geometrical model in order to understand the dynamics of these Lorenz attractors. In Fig. 1 we give a sketch of the geometric attractor. Moreover,

☆ Partially supported by Fondecyt-CHILE grant # 1000098 and PRONEX-BRAZIL on Dynamical Systems, Brazil.
* Corresponding author.
E-mail address: rlabarca@lauca.usach.cl (R. Labarca).

0294-1449/$ – see front matter © 2005 Elsevier SAS. All rights reserved.
doi:10.1016/j.anihpc.2005.09.001

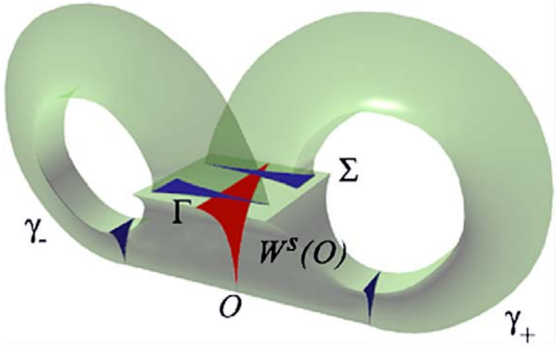

Fig. 1. Geometric Lorenz attractor.

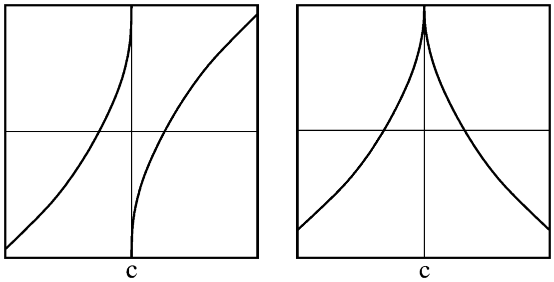

Fig. 2. One-dimensional return map.

in Fig. 2 we represent the one-dimensional models associated to the attractor. The right-hand side of this picture corresponds to the numerical experiments of Lorenz, who used a cross-section different from that of Guckenheimer and Williams, who obtained a map as in the left-hand side of the figure. The combinatorial dynamics of both one-dimensional maps sketched in this figure are equivalent. In this work we will concentrate our attention on piecewise increasing one-dimensional maps.

Using this geometrical model the dynamical behavior of the three-dimensional vector field can be reduced to the dynamical behavior of a one-dimensional map with one discontinuity and Guckenheimer and Williams, [8], used this fact to show uncountable many classes of non-equivalent geometric Lorenz attractors. The evidence of the non-equivalence were the kneading sequences associated to these one-dimensional maps. The class of one-dimensional maps defined in this way is included in the class of one-dimensional maps which we work here (see the definition of the set DM_0 given in Section 2.1).

Associated to any $f \in DM_0$ we have two kneading sequences $(a_f, b_f) = I(f)$ (see Section 2.3 for the definition of these sequences) that satisfy $a_f = \inf\{\sigma^k(a_f),\ k \in \mathbb{N}\}, b_f = \sup\{\sigma^k(b_f),\ k \in \mathbb{N}\}$ and $\{a_f, b_f\} \subset \Sigma_{a_f,b_f}$ (here $\Sigma_{a,b}$ denotes the set $\bigcap_{i=0}^{\infty} \sigma^i([a,b])$; see Section 2.3 for details). These properties inspired (see [22]) the following definitions. A sequence of two symbols $a = (0,\ldots) \in \Sigma_2$ (resp. $b = (1,\ldots) \in \Sigma_2$) is called *minimal* (resp. *maximal*) if $a = \inf\{\sigma^k(a),\ k \in \mathbb{N}\}$ (resp. $b = \sup\{\sigma^k(b),\ k \in \mathbb{N}\}$. Here $\sigma : \Sigma_2 \to \Sigma_2$ denotes the usual shift map. We will denote by Min_2 (resp. Max_2) the set of minimal (resp. maximal) sequences in Σ_2. These two properties allow us to define the *Lexicographical World* as $LW = \{(a,b) \in \mathrm{Min}_2 \times \mathrm{Max}_2,\ \{a,b\} \subset \Sigma_{a,b}\}$. The itinerary

$$I : DM_0 \to LW, \qquad f \to (a_f, b_f)$$

defines a continuous and surjective map (see Section 2.5). We will say that the map $f \in DM_0$ has *essentially* the same dynamics as $g \in DM_0$ if $I(f) = I(g)$. It is clear that two topologically equivalent maps are essentially equivalent.

Notice that one of the main topological obstructions for two maps in DM_0 with essentially the same dynamics being conjugated is the presence of wandering intervals for some of these maps. We prove in Proposition 1 that generic C^2 maps in DM_0 do not have non-trivial wandering intervals.

Therefore, the lexicographical world provide a *universal model* for (essentially) equivalent dynamics in this context. This means the following: given $(a,b) \in LW$, there is $f \in DM_0$ such that $\Sigma_{a,b} = \Sigma_{a_f,b_f}$ and a surjective map $I_f : \Gamma_f \to \Sigma_{a,b}$ such that $I_f \circ f = \sigma \circ I_f$ and reciprocally (see Section 2.3 for the definition of the set Γ_f, the definition of the map I_f and the proof of the realization lemma). Clearly, if we are able to describe the different dynamics present in this universal model then we are able to prove some dynamical properties of the elements in DM_0. Also our set, DM_0, contains the topologically expanding Lorenz maps as defined in [9]. The kneading sequences, $(a,b) \in LW$ associated to expanding maps satisfies the condition $a \leqslant \sigma^n(a) < b$ and $a < \sigma^n(b) \leqslant b$, $\forall n \geqslant 0$. Let denote by $TE \subset LW$ this set. One of the problems posed in [9] is to describe the set TE. In Section 4 we characterize the local fractal properties of TE (and LW).

In these directions are the main results of the present paper: we describe some metric and geometrical properties of the lexicographical world which "essentially" represents all the allowed dynamics for piecewise continuous increasing maps and we use these properties to establish some results for the elements in DM_0. For instance, among other results, we prove