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A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations

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Abstract

We present a new definition of the viscosity solution for a class of integro-differential equations in a bounded open domain Ω in $\mathbf{R}^N$. We consider either the Dirichlet boundary condition or the Neumann boundary condition on the boundary. These equations correspond to the process which is a combination of the jumps and the Brownian motion. We give the comparison and the existence results for the viscosity solution in our new framework.

Résumé

Nous présentons une nouvelle définition de la solution de viscosité pour certaine classe des équations intégro-différentielles sur un ouvert borné dans $\mathbf{R}^N$. Sur le bord, on considère soit la condition de Dirichlet ou soit la condition de Neumann. Ces équations correspondent au processus de la combinaison des sauts et du mouvement brownien. Nous donnons des résultats de la comparaison et de l'existence de la solution de viscosité dans notre cadre.

Keywords: Integro-differential equations; Lévy operator; Viscosity solutions; Existence and uniqueness results

1. Introduction

We study the comparison and the uniqueness of viscosity solutions for the following classes of partial-integral differential equations.

(Stationary problem)

$$F\bigl(x, u, \nabla u, \nabla^2 u\bigr) - \sup_{\alpha\in\mathcal{A}}\left\{-\int_{\mathcal{P}} u\bigl(x+\beta(x,z,\alpha)\bigr) - u(x) - \bigl\langle\beta(x,z,\alpha), \nabla u(x)\bigr\rangle q(\mathrm{d}z)\right\} = 0 \quad \text{in } \Omega. \tag{1}$$

(Evolutionary problem)

$$\frac{\partial u}{\partial t}(x,t) + F\bigl(x, u, \nabla u, \nabla^2 u\bigr)$$

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$$-\sup_{\alpha\in\mathcal{A}}\left\{-\int_{\mathcal{P}} u\big(x+\beta(x,z,\alpha),t\big)-u(x,t)-\big\langle\beta(x,z,\alpha),\nabla u(x,t)\big\rangle q(\mathrm{d}z)\right\}=0 \quad \text{in } \Omega\times(0,T), \tag{2}$$

$$u(x,0)=u_0(x) \quad \text{on } \Omega. \tag{3}$$

Here, Ω is a bounded smooth open subset of $\mathbf{R}^N$, u, ∇u, $\nabla^2 u$ denote respectively a scalar function on Ω, the gradient and the Hessian of u, and $\langle\,,\rangle$ stands for the scalar product in $\mathbf{R}^N$. The set $\mathcal{A}$ is a compact metric space, β is an $\mathbf{R}^N$-valued function defined on $\Omega\times\mathbf{R}^n\times\mathcal{A}$ such that

$$\beta_1(x)|z|\leqslant\big|\beta(x,z,\alpha)\big|\leqslant\beta_2(x)|z|, \quad \big|\beta(x,z,\alpha)-\beta(y,z,\alpha)\big|\leqslant L|x-y||z|$$
$$\forall(x,y)\in\Omega\times\Omega,\ \forall(z,\alpha)\in\mathbf{R}^n\times\mathcal{A}, \tag{4}$$

where $\beta_1(x)\geqslant 0$, $\beta_2(s)>0$ are continuous functions defined in Ω and $L>0$ is a constant, $\mathcal{P}=\mathcal{P}(x,\alpha)$ ($(x,\alpha)\in\Omega\times\mathcal{A}$) is a subset of $\mathbf{R}^n$ defined by

$$\mathcal{P}(x,\alpha)=\big\{z \mid z\in\mathbf{R}^n,\ x+\theta\beta(x,z,\alpha)\in\Omega\ \forall 0\leqslant\theta\leqslant 1\big\}, \tag{5}$$

where we remark that from (4) for some $\varepsilon>0$ small enough $U_\varepsilon(0)=\{|z|\leqslant\varepsilon\}\subset\mathcal{P}(x,\alpha)$. The so-called Lévy measure $q(\mathrm{d}z)$ is a positive Radon measure such that

$$\int_{|z|\leqslant 1}|z|^2 q(\mathrm{d}z)+\int_{|z|>1}1q(\mathrm{d}z)<\infty \tag{6}$$

holds. A given continuous function $F(x,r,p,Q)$ defined on $\Omega\times\mathbf{R}\times\mathbf{R}^N\times\mathbf{S}^N$ satisfies the following conditions.

(Properness and degenerate ellipticity:)

$$F(x,r,p,X)\leqslant F(x,s,p,Y) \quad \text{if } r\leqslant s,\ Y\leqslant X. \tag{7}$$

On the boundary $\partial\Omega$, we consider Dirichlet and Neumann type boundary conditions.

(Dirichlet B.C.)

$$u=\phi \quad \text{on } \partial\Omega, \tag{8}$$

where ϕ is a given continuous function defined on $\partial\Omega$ in the case of (1), and on $\partial\Omega\times(0,T)$ in the case of (2).

(Neumann B.C.)

$$\langle\nabla u,\mathbf{n}\rangle=0 \quad \text{on } \partial\Omega, \tag{9}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

The goal of this paper is to give the comparison and the existence of solutions for the above integro-differential equations. For this purpose, we propose the definitions of viscosity solutions, which is a little bit different from the ones in the preceding works. We refer the readers papers of Gimbert and Lions [11], Sayah [17], Alvarez and Tourin [2], Barles, Buckdahn and Pardoux [5], Jacobson and Karlsen [13], Imbert [12], etc. for the treatment of problems similar to (1) and (2) by the viscosity solutions theory. In this paper, differently from the former works we use the second-order superjet (resp. subjet) of an upper semi-continuous function $u\in\mathrm{USC}(\Omega)$ (resp. lower semi-continuous function $v\in\mathrm{LSC}(\Omega)$) in the non-local term of the equation, in particular around the origin ($\{|z|\leqslant\varepsilon\}$) to get rid of the singularity of the Lévy measure. From this reason, we need not only the superjet of u (resp. subjet of v) at a point $\hat{x}\in\Omega$, i.e. $(p,X)\in J_{\Omega}^{2,+}u(\hat{x})$ (resp. $J_{\Omega}^{2,-}v(\hat{x})$), but also the second-order remainders. If $(p,X)\in J_{\Omega}^{2,+}u(\hat{x})$ (resp. $J_{\Omega}^{2,-}v(\hat{x})$), then for any $\delta>0$ there exists $\varepsilon>0$ such that

$$u(x)\leqslant u(\hat{x})+\langle p,x-\hat{x}\rangle+\frac{1}{2}\big\langle X(x-\hat{x}),x-\hat{x}\big\rangle+\delta|x-\hat{x}|^2 \quad \text{if } |x-\hat{x}|\leqslant\varepsilon$$

(resp.

$$v(x)\geqslant v(\hat{x})+\langle p,x-\hat{x}\rangle+\frac{1}{2}\big\langle X(x-\hat{x}),x-\hat{x}\big\rangle-\delta|x-\hat{x}|^2 \quad \text{if } |x-\hat{x}|\leqslant\varepsilon).$$

To interpret this to the problem (1), from (4) for any $(\hat{x},\alpha)\in\Omega\times\mathcal{A}$, and any $\delta>0$ there exists $\varepsilon>0$ such that

$$U_\varepsilon=\big\{z \mid z\in\mathbf{R}^n,\ |z|\leqslant\varepsilon\big\}\subset\mathcal{P}(\hat{x},\alpha) \tag{10}$$