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C^1-regularity of the Aronsson equation in $\mathbf{R}^2$

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Abstract

For a nonnegative, uniformly convex $H \in C^2(\mathbf{R}^2)$ with $H(0) = 0$, if $u \in C(\Omega)$, $\Omega \subset \mathbf{R}^2$, is a viscosity solution of the Aronsson equation (1.7), then $u \in C^1(\Omega)$. This generalizes the C^1-regularity theorem on infinity harmonic functions in $\mathbf{R}^2$ by Savin [O. Savin, C^1-regularity for infinity harmonic functions in dimensions two, Arch. Ration. Mech. Anal. 176 (3) (2005) 351–361] to the Aronsson equation.

Résumé

Si $H \in C^2(\mathbf{R}^2)$ est une fonction uniformément convexe telle que $H(0) = 0$, et si $u \in C(\Omega)$, $\Omega \subset \mathbf{R}^2$, est une solution de viscosité de l'équation d'Aronsson (1.7), alors $u \in C^1(\Omega)$. Ceci généralise à l'équation d'Aronsson le théorème de C^1-régularité de Savin [O. Savin, C^1-regularity for infinity harmonic functions in dimensions two, Arch. Ration. Mech. Anal. 176 (3) (2005) 351–361] pour les fonctions ∞-harmoniques dans $\mathbf{R}^2$.

MSC: 35J60; 35J70

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1. Introduction

Calculus of variations in L^∞ was initiated by Aronsson [1–4] in 1960s. Thanks to both the development of theory of viscosity solutions of elliptic equations by Crandall and Lions (cf. [17]) and several applications to applied fields (cf. Barron [8], Barron and Jensen [9], Aronsson, Crandall and Juutinen [7], and Crandall [13]), there have been great interests in the last few years to study the minimization problem of the *supremal* functional:

$$F(u, \Omega) = \operatorname*{ess\,sup}_{x \in \Omega} H\big(x, u(x), \nabla u(x)\big), \quad \Omega \subset \mathbf{R}^n,\ u \in W^{1,\infty}\big(\Omega, \mathbf{R}^l\big). \tag{1.1}$$

Barron, Jensen and Wang [10] have established both necessary and sufficient conditions for the sequentially lower semicontinuity property of the supremal functional F in $W^{1,\infty}$, which are suitable L^∞-versions of Morrey's qua-

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siconvexity (cf. [20]) for integral functionals. In the scalar case ($l = 1$), Barron, Jensen and Wang [11] (see also Crandall [12]) have established, under appropriate conditions, the existence of *absolute minimizers* and proved that any absolute minimizer is a viscosity solution of the Aronsson equation

$$-\nabla_x\big(H\big(x, u(x), \nabla u(x)\big)\big) \cdot H_p\big(x, u(x), \nabla u(x)\big) = 0, \quad x \in \Omega. \tag{1.2}$$

Among other results in [22], the second author has showed that the convexity of $H(\cdot, p)$ are sufficient for viscosity solutions of the Aronsson equation (1.2) to be absolute minimizers of F.

Partially motivated by [22] and Crandall, Evans and Gariepy [16], Gariepy, Wang and Yu [18] have established the equivalence between absolute minimizers of F and viscosity solutions of Aronsson equation (1.2) for quasiconvex Hamiltonians $H = H(p) \in C^2(\mathbf{R}^n)$ by introducing the comparison principle of *generalized cones* (see Theorem 2.1 in Section 2 below).

In this paper, we are mainly interested in the regularity (e.g. differentiability or C^1) of viscosity solutions of the Aronsson equation.

Before stating the main results, we would like to review some of previous results for $H(p) = |p|^2$, $p \in \mathbf{R}^n$. It is well known (cf. [5,6,19]) that Eq. (1.2) is the infinity-Laplace equation, of which a viscosity solution is called an infinity harmonic function,

$$-\Delta_\infty u := -\sum_{i,j=1}^{n} u_i u_j u_{ij} = 0, \quad \text{in } \Omega, \tag{1.3}$$

and the absolute minimality is called absolute minimal Lipschitz extension (or AMLE) property:

for any open subset $U \Subset \Omega$ and $v \in W^{1,\infty}(U) \cap C(\bar{U})$ with $v = u$ on ∂U, *we have*

$$\|\nabla u\|_{L^\infty(U)} \leqslant \|\nabla v\|_{L^\infty(U)}. \tag{1.4}$$

Aronsson [5] proved that any C^2-infinity harmonic function satisfies the AMLE property. Jensen [19] has proved the equivalence between an infinity harmonic function and the AML property, and the unique solvability of the Dirichlet problem of Eq. (1.3).

Crandall, Evans and Gariepy [16] have recently showed that $u \in C^0(\Omega)$ is an infinity harmonic function iff u enjoys comparison with cones in Ω:

for any open subset $U \Subset \Omega$, $a, b \in \mathbf{R}$, and $x_0 \in \Omega$,

$$u(x) \leqslant (\geqslant) a + b|x - x_0| \quad \text{on } \partial\big(U \setminus \{x_0\}\big) \Rightarrow u(x) \leqslant (\geqslant) a + b|x - x_0| \quad \text{in } U. \tag{1.5}$$

In a very recent important paper [21], Savin has utilized [16] and Crandall and Evans [15] to prove that *any C^0-infinity harmonic function in $\Omega \subset \mathbf{R}^2$ is in $C^1(\Omega)$.*

In this paper, we extend the main theorems of [21] to the Aronsson equation for a class of Hamiltonian functions $H \in C^2(\mathbf{R}^2)$.

First we extend [15] and obtain the following theorem on blow-up limits of viscosity solutions of the Aronsson equation on $\mathbf{R}^n$ for $n \geqslant 2$, which may have its own interests.

Theorem A. *Assume that $H \in C^2(\mathbf{R}^n)$ is nonnegative and uniformly convex, i.e. there is $\alpha_H > 0$ such that*

$$p^T \cdot H_{pp}(p) \cdot p \geqslant \alpha_H |p|^2, \quad \forall p \in \mathbf{R}^n, \tag{1.6}$$

and $H(0) = 0$. Suppose that $u \in C^0(\Omega)$, $\Omega \subset \mathbf{R}^n$, is a viscosity solution of the Aronsson equation:

$$-H_p\big(\nabla u(x)\big) \otimes H_p\big(\nabla u(x)\big) : \nabla^2 u(x) = 0, \quad \text{in } \Omega, \tag{1.7}$$

then for any $x \in \Omega$, there exists a $e_{x,r} \in \mathbf{R}^n$, with $H(e_{x,r}) = S^+(H, u, x)$ (see Section 2 for the definition of $S^+(H, u, x)$), such that

$$\lim_{r \to 0} \max_{B_r(x)} \frac{|u(y) - u(x) - e_{x,r} \cdot (y - x)|}{r} = 0. \tag{1.8}$$

Based on Theorem A and the main theorem of [18], we are able to make necessary modifications of the idea of [21] to prove