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Singular limits for a 4-dimensional semilinear elliptic problem with exponential nonlinearity

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Abstract

Using some nonlinear domain decomposition method, we prove the existence of branches of solutions having singular limits for some 4-dimensional semilinear elliptic problem with exponential nonlinearity.

Résumé

En utilisant une variante non linéaire de la méthode de décomposition de domaines, nous démontrons l'existence de branches de solutions ayant une limite singulière, pour une équation semilinéaire elliptique avec nonlinéarité exponentielle, en dimension 4.

Keywords: Singular perturbations; Concentration phenomena

1. Introduction and statement of the results

In the last decade important work has been devoted to the understanding of singularly perturbed problems, mostly in a variational framework. In general, a Liapunov–Schmidt type reduction argument is used to reduce the search of solutions of singularly perturbed nonlinear partial differential equations to the search of critical points of some function that is defined over some finite dimensional domain.

One of the purposes of the present paper is to present a rather efficient method to solve such singularly perturbed problems. This method has already been used successfully in geometric context (constant mean curvature surfaces, constant scalar curvature metrics, extremal Kähler metrics, manifolds with special holonomy, ...) but has never appeared in the framework of nonlinear partial differential equations. We felt that, given the interest in singular perturbation problems, it was worth illustrating this method on the following model problem:

Assume that $\Omega \subset \mathbb{R}^4$ is a regular bounded open domain in $\mathbb{R}^4$. We are interested in positive solutions of

$$\begin{cases} \Delta^2 u = \rho^4 \mathrm{e}^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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when the parameter ρ tends to 0. Obviously, the application of the implicit function theorem yields, for ρ close to 0, the existence of a smooth one parameter family of solutions $(u_\rho)_\rho$ that converges uniformly to 0 as ρ tends to 0. This branch of solutions is usually referred to as the branch of *minimal solutions* and there is by now quite an important literature that is concerned with the understanding of this particular branch of solutions [12].

The problem we would like to consider is the existence of other branches of solutions as ρ tends to 0. To describe our result, let us denote by $G(x,\cdot)$ the solution of

$$\begin{cases} \Delta^2 G(x,\cdot) = 64\pi^2\delta_x & \text{in } \Omega, \\ G(x,\cdot) = \Delta G(x,\cdot) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

It is easy to check that the function

$$R(x,y) := G(x,y) + 8\log|x-y| \tag{3}$$

is a smooth function. Finally, we define

$$W\bigl(x^1,\dots,x^m\bigr) := \sum_{j=1}^m R\bigl(x^j,x^j\bigr) + \sum_{j\neq\ell} G\bigl(x^j,x^\ell\bigr). \tag{4}$$

Our main result reads:

Theorem 1.1. *Assume that* $(x^1,\dots,x^m)$ *is a* nondegenerate *critical point of* W*, then there exist* $\rho_0>0$ *and* $(u_\rho)_{\rho\in(0,\rho_0)}$*, a one parameter family of solutions of* (1)*, such that*

$$\lim_{\rho\to 0} u_\rho = \sum_{j=1}^m G(x^j,\cdot)$$

in $C^{4,\alpha}_{\mathrm{loc}}(\Omega - \{x^1,\dots,x^m\})$.

This result is in agreement with the result of Lin and Wei [6] where sequences of solutions of (1) that blow up as ρ tends to 0 are studied. Indeed, in this paper, the authors show that blow up points can only occur at critical points of W.

Our result reduces the study of nontrivial branches of solutions of (1) to the search of critical points of the function W defined in (4). Observe that the assumption on the nondegeneracy of the critical point is a rather mild assumption since it is certainly fulfilled for generic choice of the regular bounded open domain Ω.

Semilinear equations involving fourth order elliptic operator and exponential nonlinearity appear naturally in conformal geometry and in particular in the prescription of the so called Q-curvature on 4-dimensional Riemannian manifolds [2,3]

$$Q_g = \frac{1}{12}\bigl(-\Delta_g S_g + S_g^2 - 3|\mathrm{Ric}_g|^2\bigr)$$

where Ric_g denotes the Ricci tensor and S_g is the scalar curvature of the metric g. Recall that the Q-curvature changes under a conformal change of metric

$$g_w = \mathrm{e}^{2w} g,$$

according to

$$P_g w + 2Q_g = 2Q_{g_w}\mathrm{e}^{4w} \tag{5}$$

where

$$P_g := \Delta_g^2 + \delta\left(\frac{2}{3}S_g I - 2\,\mathrm{Ric}_g\right)d \tag{6}$$

is the Paneitz operator, which is an elliptic 4th order partial differential operator [3] and which transforms according to

$$\mathrm{e}^{4w} P_{\mathrm{e}^{2w}g} = P_g, \tag{7}$$