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Analysis of boundary bubbling solutions for an anisotropic Emden–Fowler equation

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Abstract

We consider the following anisotropic Emden–Fowler equation

$$\nabla(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and a is a positive smooth function. We study here the phenomenon of boundary bubbling solutions which *do not exist* for the isotropic case $a \equiv constant$. We determine the localization and asymptotic behavior of the boundary bubbles, and construct some boundary bubbling solutions. In particular, we prove that if $\bar{x} \in \partial\Omega$ is a strict local minimum point of a, there exists a family of solutions such that $\varepsilon^2 a(x)e^u\,dx$ tends to $8\pi a(\bar{x})\delta_{\bar{x}}$ in $\mathcal{D}'(\mathbb{R}^2)$ as $\varepsilon \to 0$. This result will enable us to get a new family of solutions for the isotropic problem $\Delta u + \varepsilon^2 e^u = 0$ in rotational torus of dimension $N \geqslant 3$.

Keywords: Boundary bubble; Blow-up analysis; Localized energy method

1. Introduction

The classical Emden–Fowler equation, or Gelfand equation

$$\Delta u + \varepsilon^2 e^u = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

has motivated a lot of studies, because it has both geometrical and physical background. When $N = 2$, (1) or more generally Eq. (2) below relates to the geometric problem of Riemannian surfaces with prescribed Gaussian curvature (see [7] and references therein). For $N \geqslant 3$, it arises in the theory of thermionic emission, isothermal gas sphere, gas combustion. It is also considered in relation with Onsager's formulation in statistical mechanics, the Keller–Segel system of chemotaxis, Chern–Simon–Higgs gauge theory and many other physical applications (see [4–6,11,13,19, 17,25] and the references therein).

It is well known that there exists a critical value $\varepsilon^* > 0$ such that when $\varepsilon > \varepsilon^*$, no solution of (1) exists while for $\varepsilon \in (0, \varepsilon^*)$, we have a family of minimal solutions which tend uniformly to zero as $\varepsilon \to 0$. When $N = 2$, for any

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$\varepsilon \in (0, \varepsilon^*)$, we have also a second solution which is nonstable and blows up as $\varepsilon \to 0$. The asymptotic behavior of nonstable solutions to (1), or to a more general equation

$$\Delta u + \varepsilon^2 k(x) e^u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u = 0 \quad \text{on } \partial\Omega \tag{2}$$

where $k(x)$ is a positive smooth function has been studied in [3,14,15,18,21,27]. Let G_D denote the standard Green's function of $-\Delta$ with Dirichlet boundary condition and H_D denote the regular part of G_D, i.e.

$$H_D(x, y) = G_D(x, y) + \frac{1}{2\pi} \log|x - y|. \tag{3}$$

If u_ε is a family of solutions to (2) satisfying

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon}\, dx \to \ell,$$

as $\varepsilon \to 0$ and $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Omega)} = \infty$, then up to a subsequence, there holds either $\ell = \infty$, $u_\varepsilon \to \infty$ for all $x \in \Omega$; or $\ell = 8\pi m$, $m \in \mathbb{N}^*$ and u_ε makes m points *simple* blow-up on $\mathcal{S} = \{x_1, \ldots, x_m\} \subset \Omega$ such that

$$\varepsilon^2 k(x) e^{u_\varepsilon}\, dx \to 8\pi \sum_{j=1}^m \delta_{x_j}, \quad u_\varepsilon \to 8\pi \sum_{j=1}^m G_D(\cdot, x_j) \quad \text{in } C^k_{\mathrm{loc}}(\overline{\Omega} \setminus \mathcal{S}),\ \forall k \in \mathbb{N},$$

where $(x_1, \ldots, x_m)$ is a critical point of Ψ defined by

$$\Psi(x) = \sum_{j=1}^m H_D(x_j, x_j) + \sum_{i\neq j} G_D(x_i, x_j) + 2\sum_{j=1}^m \log k(x_j).$$

Conversely, many authors have constructed blow-up solutions, see for example [2,9,10,20]. So the solutions of Eq. (1) or (2) are now well understood in dimension two.

Here we consider the following generalized Emden–Fowler equation

$$\Delta_a u + \varepsilon^2 e^u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad u = 0 \quad \text{on } \partial\Omega \tag{4}$$

where Ω is a smooth bounded domain, Δ_a is the operator

$$\Delta_a u = \frac{1}{a(x)} \nabla\big[a(x)\nabla u\big] = \Delta u + \nabla \log a \nabla u$$

and $a(x)$ is a smooth function over Ω satisfying

$$0 < a_1 \leqslant a(x) \leqslant a_2 < +\infty. \tag{5}$$

Our motivation is due to the fact that few is known for Eq. (1) in dimension $N \geqslant 3$. As far as we know, the only explicit results in higher dimensions concern the radial solutions in spheres (see [11,13]) or in annuli (see [22]). It is worth to mention that Pacard proved in [23] (see also [16]) that for annuli, i.e. $\Omega = A_{r_0} = \{x \in \mathbb{R}^N, r_0 < \|x\| < 1\}$, there exists $X \subset (0, 1)$ of measure equal to 1 such that for all $r_0 \in X$, there are infinitely many symmetry breaking points with bifurcation from the branch of radial solutions. Unfortunately, we do not have no more precise information about the behavior of these nonradial solutions. However, through these results, we observe already a quite different situation with the case in dimension two. Our idea here is to consider axially symmetric solutions of (1) in a torus, and try to give some precise descriptions of them. In fact, let $\mathbb{T}$ be a standard N-dimensional torus ($N \geqslant 3$), i.e.

$$\mathbb{T} = \Big\{x = (x_i) \in \mathbb{R}^N;\ \Big(\sqrt{x_1^2 + \cdots + x_{N-1}^2} - 1\Big)^2 + x_N^2 \leqslant r_0^2\Big\} \tag{6}$$

with $0 < r_0 < 1$. If we look for solutions of (1) in the form of $u(x) = u(r, s)$ where

$$r = \sqrt{x_1^2 + \cdots + x_{N-1}^2} \quad \text{and} \quad s = x_N,$$

a direct calculus yields that the problem (1) is transformed to

$$\nabla(r^{N-2}\nabla u) + \varepsilon^2 r^{N-2} e^u = 0 \quad \text{in } \Omega_{\mathbb{T}}, \qquad u = 0 \quad \text{on } \partial\Omega_{\mathbb{T}}, \tag{7}$$