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Ann. I. H. Poincaré – AN 23 (2006) 911–927

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Large time behavior for nonlinear higher order convection–diffusion equations

Comportement asymptotique pour des equations de convection–diffusion nonlinéaires d'ordre supérieur

Mahmoud Qafsaoui

Ecole Supérieure des Techniques Aéronautiques et de Construction Automobile, 34, rue Victor Hugo, 92300 Levallois Perret, France

Received 11 January 2005; received in revised form 5 September 2005; accepted 22 September 2005

Available online 9 February 2006

Abstract

We study the large time asymptotic behavior, in L^p $(1 \leqslant p \leqslant \infty)$, of higher derivatives $D^\gamma u(t)$ of solutions of the nonlinear equation

$$\begin{cases} u_t + \mathcal{T}u = a \cdot \nabla^\theta(\psi(u)) & \text{on } \mathbb{R}^n \times (0, \infty), \\ u(0) = u_0 \in L^1(\mathbb{R}^n), \end{cases} \tag{1}$$

where the integers n and θ are bigger than or equal to 1, a is a constant vector in $\mathbb{R}^p$ with $p = \binom{\theta+n-1}{n-1} = \frac{(\theta+n-1)!}{\theta!(n-1)!}$. The function ψ is a nonlinearity such that $\psi \in \mathcal{C}^\theta(\mathbb{R})$ and $\psi(0) = 0$, and $\mathcal{T}$ is a higher order elliptic operator with nonsmooth bounded measurable coefficients on $\mathbb{R}^n$. We also establish faster decay when $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Résumé

Nous étudions le comportement asymptotique, dans L^p $(1 \leqslant p \leqslant \infty)$, des dérivées d'ordre supérieur $D^\gamma u(t)$ des solutions de l'équation nonlinéaire (1), où $n \in \mathbb{N}^*$, $\theta \in \mathbb{N}^*$ et a est un vecteur constant de $\mathbb{R}^p$ avec $p = \binom{\theta+n-1}{n-1}$. La fonction ψ est nonlinéaire vérifiant $\psi \in \mathcal{C}^\theta(\mathbb{R})$ et $\psi(0) = 0$, et $\mathcal{T}$ est un opérateur elliptique d'ordre supérieur à coefficients peu réguliers dans $\mathbb{R}^n$. Nous étudions également le cas particulier où $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

MSC: 35B40; 35K25; 35K57; 35G05; 35A08

Keywords: Generalized convection–diffusion equations; Higher order operators; Heat kernel; Asymptotic behavior; L^p-estimates

Mots-clés : Équations de convection–diffusion généralisées ; Noyau de la chaleur ; Opérateurs d'ordre supérieur ; Comportement asymptotique ; Estimations-L^p

E-mail address: mqafsaoui@estaca.fr (M. Qafsaoui).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2005.09.008

1. Introduction

Our aim is to study the asymptotic behavior of higher derivatives of solutions of the Cauchy problem for the generalized convection–diffusion equation (1) using sufficiently smooth nonlinearities ψ. A typical example of (1) is given by

$$\begin{cases} u_t + \big(\Delta^m A \Delta^m\big)u = a \cdot \nabla^\theta\big(\psi(u)\big) & \text{on } \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 \in L^1\big(\mathbb{R}^n\big), \end{cases}$$

where A is a bounded measurable positive function independent of time t.

Let us start by mentioning some works which inspired ours. Escobedo and Zuazua studied in [3] the large time behavior of solutions of the Cauchy problem for the convection–diffusion equation (1) with $\mathcal{T} = -\Delta$, $\psi(u) = |u|^{q-1}u$ and $\theta = 1$. They proved, for $q > 1$, the existence and the uniqueness of a classical solution $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^n))$ such that $u \in \mathcal{C}((0,\infty); W^{2,p}(\mathbb{R}^n)) \cap \mathcal{C}^1((0,\infty); L^p(\mathbb{R}^n))$ for every $p \in (1,\infty)$. The argument used relies essentially on the classical Banach fixed point theorem. They also showed that, for t large, this solution behaves like the heat kernel K_t which can be regarded as the fundamental solution of the heat equation with the Dirac mass as initial data. More precisely, under the assumption $q > 1 + 1/n$, if M denotes the mass of u_0 ($M = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x$) then the solution u satisfies for all $p \in [1,\infty]$,

$$\lim_{t\to+\infty} t^{\frac{n}{2}(1-\frac{1}{p})} \big\|u(t) - MK_t\big\|_p = 0.$$

They also obtained a faster decay in the particular case when the initial data u_0 belongs to $L^1(\mathbb{R}^n; 1+|x|) \cap L^q(\mathbb{R}^n)$. The techniques used rely on standard heat kernel estimates on the integral representation of the solution. Subsequently, they finished their work by an extension to the more general equation $u_t - \Delta u = a \cdot \nabla(\psi(u))$ where ψ is an arbitrary sufficiently smooth nonlinearity.

In [2], Biller, Karch and Woyczyński studied the large time behavior of solutions of the Lévy conservation laws $u_t + \mathcal{L}u + \nabla \cdot \psi(u) = 0$ with initial data u_0, where ψ is a nonlinearity and $(-\mathcal{L})$ is the generator of a positivity-preserving symmetric Lévy semigroup on $L^1(\mathbb{R}^n)$. In particular, they showed that in the case where the symbol $\mathcal{A}$ of the operator $\mathcal{L}$ satisfies $\mathcal{A}(\zeta) \sim |\zeta|^\iota$ for $|\zeta| < 1$ $(0 < \iota < 2)$ and $\mathcal{A}(\zeta) \sim |\zeta|^2$ for $|\zeta| > 1$, and under the assumptions $\psi \in \mathcal{C}^2$ with $\psi'(0) = 0$ and $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, the following holds

$$\lim_{t\to+\infty} t^{\frac{n}{\iota}(1-\frac{1}{p})} \big\|u(t) - \mathrm{e}^{-t\mathcal{L}}u_0\big\|_p = 0 \quad \text{for every } p \in [1,\infty]$$

as for the corresponding linear equation, and if $F := \int_0^\infty \int_{\mathbb{R}^n} \psi(u(y,s))\,\mathrm{d}y\,\mathrm{d}s$,

$$\lim_{t\to+\infty} t^{\frac{n}{\iota}(1-\frac{1}{p})+\frac{1}{\iota}} \big\|u(t) - \mathrm{e}^{-t\mathcal{L}}u_0 + F \cdot \big(\nabla \mathrm{e}^{-t\mathcal{L}}\big)\big\|_p = 0 \quad \text{for every } p \in (1,\infty]$$

resulting from the presence of the nonlinear term.

In [4], we considered the equation

$$\begin{cases} u_t + \mathcal{L}_t u = a\nabla u & \text{on } \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 \in L^1\big(\mathbb{R}^n\big), \end{cases} \tag{2}$$

where $\mathcal{L}_t = L_0^* b L_0$ (see below for the definition of L_0) and b is a positive bounded function such that $b(x,t) = b(x+at)$. We showed that the derivatives $D^\gamma u$ of order less than or equal to $2m-1$ of the solution to (2) have an asymptotic behavior similar to the one of the corresponding derivatives of the heat kernel with speed a,

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} \big\|D_x^\gamma u(x,t) - MD_x^\gamma K_t(x+at)\big\|_p = 0, \tag{3}$$

for all $p \in [1,\infty]$. The method used to derive this result was the simple change of variables $v(x,t) := u(x-at,t)$ which reduces the study to the heat equation

$$\begin{cases} v_t + \mathcal{L}_0 v = 0 & \text{on } \mathbb{R}^n \times (0,\infty), \\ v(0) = v_0 = u_0. \end{cases}$$