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Approximation of solutions to multidimensional parabolic equations by approximate approximations

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ABSTRACT

We propose a fast method for high order approximations of the solution of n-dimensional parabolic problems over hyper-rectangular domains in the framework of the method of approximate approximations. This approach, combined with separated representations, makes our method effective also in very high dimensions. We report on numerical results illustrating that our formulas are accurate and provide the predicted approximation rate 6 also in high dimensions.

1. Introduction

Multidimensional boundary value problems arise in mathematical physics, financial mathematics, biology, chemistry and other applied fields. The computational complexity of the algorithms grows exponentially in the dimension. This effect was called "curse of dimensionality" (Bellmann) and it was the greatest impediment to solving real-world problems.

In [1] and [2], Beylkin and Mohlenkamp introduced the strategy of "separated representations" (also tensor structured approximations) which allowed to perform numerical computations in higher dimensions.

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In recent years modern methods based on tensor product approximations have been applied successfully (e.g. [3–10] and the references therein) to some class of multidimensional integral operators.

Some algorithms approximate the operator kernel by a linear combination of exponentials or Gaussians leading to a tensor product approximation. Other methods are based on piecewise polynomial approximations of a separated representation of the density. Then the integral operator applied to the basis functions is approximated by computing a number of one-dimensional integrals.

A different method with high accuracy, which does not approximate or modify the kernel of the integral operator, was introduced in [11] and [12] for the cubature of high dimensional Newton potential over the full space and over half spaces. Here the integral density is approximated by basis functions introduced in the method of approximate approximations, which provides high order semi-analytic cubature formulas. This approach, combined with separated representations, makes the method fast and effective also in very high dimensions. The new approach can be generalized to potentials of other elliptic differential operators acting on densities on hyper-rectangular domains. In [13] and, more generally, in [14], a corresponding cubature method was introduced for stationary advection–diffusion equations, which provides very efficiently high order approximations.

In this paper we show that our approach can be extended to parabolic problems. We propose a fast method in the framework of approximate approximations for the n-dimensional time dependent problem

$$\frac{\partial u}{\partial t} - \Delta_{\mathbf{x}} u + 2\mathbf{b}\cdot\nabla_{\mathbf{x}} u + c\,u = f(\mathbf{x},t),$$
$$u(\mathbf{x},0) = g(\mathbf{x}) \tag{1.1}$$

for $(\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+$ with $\mathbb{R}_+ = [0,\infty)$, $\mathbf{b} \in \mathbb{C}^n$, $c \in \mathbb{C}$. We suppose that f and g are supported with respect to $\mathbf{x}$ in a hyper-rectangle $[\mathbf{P},\mathbf{Q}] = \{\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n : P_j \le x_j \le Q_j,\ j = 1,\dots,n\}$, $\operatorname{supp} f \subseteq [\mathbf{P},\mathbf{Q}]\times\mathbb{R}_+$, $\operatorname{supp} g \subseteq [\mathbf{P},\mathbf{Q}]$. The solution of (1.1) can be written as [15, p. 49]

$$u(\mathbf{x},t) = \mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} f(\mathbf{x},t) + \mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} g(\mathbf{x},t)\,, \tag{1.2}$$

where

$$\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} g(\mathbf{x},t) = \frac{e^{-ct}}{(4\pi\,t)^{n/2}} \int\limits_{[\mathbf{P},\mathbf{Q}]} e^{-|\mathbf{x}-\mathbf{y}-2\mathbf{b}t|^2/(4t)}\, g(\mathbf{y})d\mathbf{y}\,, \tag{1.3}$$

$$\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} f(\mathbf{x},t) = \int\limits_0^t \frac{e^{-cs}ds}{(4\pi\,s)^{n/2}} \int\limits_{[\mathbf{P},\mathbf{Q}]} e^{-|\mathbf{x}-\mathbf{y}-2\mathbf{b}s|^2/(4s)}\, f(\mathbf{y},t-s)d\mathbf{y}$$
$$= \int\limits_0^t (\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} f(\cdot,s))(\mathbf{x},t-s)ds. \tag{1.4}$$

Our method consists in approximating the functions f and g via the basis functions introduced by approximate approximations, which are product of Gaussians and special polynomials. The action of the potential $\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}$ applied to the basis functions admits a separated representation, i.e., it is represented as product of functions depending only on one of the space variables. Then a separated representation of the initial condition g (see (2.9)) provides a separated representation of the potential. Moreover, the action of $\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}$ on the basis functions allows for one-dimensional integral representations with separated integrands. This construction, combined with an accurate quadrature rule as suggested in [16] and a separated representation of the density f, provides a separated representation of the integral operator (1.4). Thus for the computation of (1.1) only one-dimensional operations are used. We derive formulas of an arbitrary high order, fast and