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Optimal adaptive ridgelet schemes for linear advection equations

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A R T I C L E   I N F O

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A B S T R A C T

In this paper we present a novel method for the numerical solution of linear advection equations, which is based on ridgelets. Such equations arise for instance in radiative transfer or in phase contrast imaging. Due to the fact that ridgelet systems are well adapted to the structure of linear transport operators, it can be shown that our scheme operates in optimal complexity, even if line singularities are present in the solution.

The key to this is showing that the system matrix (with diagonal preconditioning) is uniformly well-conditioned and compressible – the proof for the latter represents the main part of the paper. We conclude with some numerical experiments about N-term approximations and how they are recovered by the solver, as well as localisation of singularities in the ridgelet frame.

1. Introduction

In the past two decades, a wide range of multiscale systems have been introduced in Applied Harmonic Analysis, which, by virtue of being exceptionally well-adapted to different signal classes, are having an ever-increasing impact in many different fields. This development started with wavelets [9] and is continuing with ridgelets [3], curvelets [5,6,8], shearlets [21,22], contourlets [12] etc. – the latter three of which fall into the framework of so-called “parabolic molecules” [15], while all of the mentioned systems are encompassed by the even broader framework of α-molecules [16].

The above-mentioned adaptivity can be made precise in the sense that these systems are able to represent certain classes of functions optimally in terms of the error decay rate of the best N-term approximation [10] – functions with point singularities for wavelets, line singularities for ridgelets and curved singularities for parabolic molecules. Since these classes make up the fundamental phenomenological features of most images across an extremely diverse set of applications, it is perhaps not surprising, that many of the above-mentioned systems were originally investigated in view of their properties regarding image processing.

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With a certain time-lag, it is becoming apparent that these systems are also very suitable for solving partial differential equations – again, wavelets were the first in this regard, for example leading to provably optimal solvers for elliptic equations [7]. For differential equations with strong directional features – such as transport equations – it is intuitively clear that optimal solvers will need to take these features into account, however, the development of solvers based on directional systems is still in its infancy.

Following recent results [17], that ridgelets permit the construction of simple diagonal preconditioners for linear advection equations which arise in collocation-type discretisation methods for kinetic transport equations (such as radiative transport), we intend this paper (and its companion [14]) to be a first step towards establishing directional representation systems as a useful tool for solving PDEs.

1.1. Linear advection equations

In this paper, we are mainly concerned with solving the following linear advection equation, more precisely, the unidirectional *advection–reaction equation*,

$$\vec{s} \cdot \nabla u(\vec{x}) + \kappa(\vec{x}) u(\vec{x}) = f(\vec{x}) \tag{1.1}$$

which describes the stationary distribution of the unknown quantity u under absorption (κ), emission (f) and transport in direction $\vec{s}$. In Subsection 1.4, we will see how this is related to more general transport problems.

Let us assume that $\kappa(\vec{x}) \geq \gamma > 0$ and $f(\vec{x}) \in \mathbb{R}$ are known at all locations $\vec{x} \in \Omega \subset \mathbb{R}^d$. Then, the above equation allows us to find the unknown u as a function $\Omega \to \mathbb{R}$.

Typical methods for the numerical solution of (1.1) include [13, Chap. 5]

- Galerkin Least Squares: $\langle Av, Au \rangle = \langle Av, f \rangle \ \forall v \in \mathcal{V}_{\text{test}}$
- Discontinuous Galerkin Methods
- Streamline Upwinding Petrov Galerkin (SUPG) [1]

All of these suffer from the fact that the transport term $s \cdot \nabla u$ leads to ill-conditioned systems of equations (with no rigorous results about the choice/efficiency of preconditioners), and that singularities in the input data may remain in the solution.

1.2. CDD-schemes and provably optimal convergence & complexity

Taking a step back, we want to briefly recall wavelet methods for elliptic operator equations, where, in [7], the authors achieved *provably* optimal convergence rates – even in the presence of singularities – and even more, bounds for the number of floating point operations (flops) necessary to carry out the computations that are linear in the number of coefficients one wants to find (which is again the best that can be hoped for without prior knowledge of the solution/coefficients). This is in stark contrast to Finite Element Methods, whose convergence rate breaks down for uniform refinement in the presence of singularities, respectively, to adaptive FEM where convergence rates have only been proven in special cases.

Of course, wavelets do not work optimally for (1.1) – which is not H^1-elliptic, and thus leads to ill-conditioned systems – either. Heuristically, this break-down of optimality can also be explained with the fact that it takes many wavelets (which are adapted to point singularities) to resolve a singularity along a line – picture a “cliff” or compare for example Fig. 7.4.

However, with the following four ingredients, it is possible to construct such an optimal scheme also in different settings [7,11,26]: