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# Slepian spatial-spectral concentration on the ball

Zubair Khalid [a,*,1], Rodney A. Kennedy [a,1], Jason D. McEwen [b,2]

[a] *Research School of Engineering, College of Engineering and Computer Science, The Australian National University, Canberra, Australia*
[b] *Mullard Space Science Laboratory, University College London, Surrey RH5 6NT, UK*

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ABSTRACT

We formulate and solve the analog of Slepian spatial-spectral concentration problem on the three-dimensional ball. Both the standard Fourier–Bessel and also the Fourier–Laguerre spectral domains are considered since the latter exhibits a number of practical advantages such as spectral decoupling and exact computation. The Slepian spatial and spectral concentration problems are formulated as eigenvalue problems, the eigenfunctions of which form an orthogonal family of concentrated functions. Equivalence between the spatial and spectral problems is shown. The spherical Shannon number on the ball is derived, which acts as the analog of the space-bandwidth product in the Euclidean setting, giving an estimate of the number of concentrated eigenfunctions and thus the dimension of the space of functions that can be concentrated in both the spatial and spectral domains simultaneously. Various symmetries of the spatial region are considered that reduce considerably the computational burden of recovering eigenfunctions, either by decoupling the problem into smaller subproblems or by affording analytic calculations. The family of concentrated eigenfunctions forms a Slepian basis that can be used to represent concentrated signals efficiently. We illustrate our results with numerical examples and show that the Slepian basis indeed permits a sparse representation of concentrated signals.



## 1. Introduction

It is well-known that functions cannot have finite support in both the spatial (or time) and spectral (or frequency) domain at the same time [42,41]. The fundamental problem of finding and representing the functions that are optimally energy concentrated in both the time and frequency domains was solved by Slepian, Landau and Pollak in the early 1960s [42,18,19,43]. This problem, herein referred to as the *Slepian*

* Corresponding author.
*E-mail addresses:* zubair.khalid@anu.edu.au (Z. Khalid), rodney.kennedy@anu.edu.au (R.A. Kennedy), jason.mcewen@ucl.ac.uk (J.D. McEwen).

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*spatial-spectral concentration problem*, or *Slepian concentration problem* for short, gives rise to the orthogonal families of functions that are optimally concentrated in the spatial (spectral) domain and exactly limited in the spectral (spatial) domain. These families of functions and their multidimensional extensions [40,39] have been used extensively in various branches of science and engineering (e.g., signal processing [46,25], medical imaging [15], geophysics [48], climatology [47], to name a few). Notably, they have been used in linear inverse problems [13], interpolation [28,35,34], extrapolation [53] and in solving partial differential equations [5,7]. Indeed, in many scientific and engineering applications, these functions have become the preferred spatial or spectral windows for the regularisation of quadratic inverse problems of power spectral estimation from spatially limited observations [48,31].

Although the Slepian spatial-spectral concentration problem was initially formulated and solved in the Euclidean domain, generalisations for various geometries and connections to wavelet analysis have also been well-studied (e.g., [27,11,8,12,3,29,30,51,36]). We note that the Slepian concentration problem for functions defined on the two-sphere $\mathbb{S}^2$ has been thoroughly revisited and investigated [3,36,51]. The resulting orthogonal family of band-limited spatially concentrated functions have been applied for localised spectral analysis [52] and spectral estimation [10] of signals (finite energy functions) defined on the sphere. There are also many applications [22,38,37] where signals or data are defined naturally on the three-dimensional ball, or ball for short. For example, signals defined on the ball arise when observations made on the sphere are augmented with radial information, such as depth, distance or redshift. Recently, a number of signal processing techniques have been tailored and extended to deal with signals defined on the ball (e.g., [38,21,22]).

In this paper, we pose, solve and analyse the Slepian concentration problem of simultaneous spatial and spectral localisation of functions defined on the ball. By considering Slepian's quadratic (energy) concentration criterion, we formulate and solve the problems to: (1) find the band-limited functions with maximum concentration in some spatial region; and (2) find the space-limited functions with maximum concentration in some region of the spectral domain. Each problem is formulated as an eigenvalue problem, the solution of which gives the orthogonal family of functions, referred to as eigenfunctions, which are either spatially concentrated while band-limited, or spectrally concentrated while space-limited. These eigenfunctions serve as an alternative basis on the ball, which we call a Slepian basis, for the representation of a band-limited or space-limited signal. We show, and also illustrate through an example, that the representation of band-limited spatially concentrated or space-limited spectrally concentrated functions is sparse in the Slepian basis, which is the essence of the Slepian spatial-spectral concentration problem. We also derive the spherical Shannon number as an equivalent of the Shannon number in the one dimensional Slepian concentration problem [43,31], which serves as an estimate of the number of concentrated functions in the Slepian basis.

For the spectral domain characterisation of functions defined on the ball we use two basis functions: (1) spherical harmonic-Bessel functions, which arise as a solution of Helmholtz's equation in three-dimensional spherical coordinates, and are referred to as *Fourier–Bessel*[3] basis functions; and (2) spherical harmonic-Laguerre functions, which are referred to as *Fourier–Laguerre* basis functions. We consider the Fourier–Laguerre functions in addition to the standard Fourier–Bessel functions as the Fourier–Laguerre functions serve as a complete basis for signals defined on the ball, enable the decoupling of the radial and angular components of the signal, and support the exact computation of forward and inverse Fourier–Laguerre transforms [22]. We show that the eigenvalue problem to find the eigenfunctions or the Slepian basis can be decomposed into subproblems when the spatial region of interest is symmetric in nature. We consider two types of symmetric regions: (1) circularly symmetric regions; and (2) circularly symmetric and radially independent regions.

As Slepian functions on the one-dimensional Euclidean domain [42,18,19,43], and other geometries [40,27,3,30,36], have been widely useful in a diverse variety of applications, we hope that the proposed

[3] A more appropriate terminology would be spherical harmonic-Bessel basis, however we adopt the established convention of using the term Fourier to denote the spherical harmonic part.