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# A generalized Schur–Horn theorem and optimal frame completions

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ABSTRACT

The Schur–Horn theorem is a classical result in matrix analysis which characterizes the existence of positive semidefinite matrices with a given diagonal and spectrum. In recent years, this theorem has been used to characterize the existence of finite frames whose elements have given lengths and whose frame operator has a given spectrum. We provide a new generalization of the Schur–Horn theorem which characterizes the spectra of all possible finite frame completions. That is, we characterize the spectra of the frame operators of the finite frames obtained by adding new vectors of given lengths to an existing frame. We then exploit this characterization to give a new and simple algorithm for computing the optimal such completion.

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## 1. Introduction

The Schur–Horn theorem [16,25] is a classical result in matrix analysis which characterizes the existence of positive-semidefinite matrices with a given diagonal and spectrum. To be precise, let $\mathbb{F}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and let $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$ be any nonincreasing sequences of nonnegative real scalars. The Schur–Horn theorem states that there exists a positive semidefinite matrix $\mathbf{G} \in \mathbb{F}^{N\times N}$ with eigenvalues $\{\lambda_n\}_{n=1}^N$ and with $\mathbf{G}(n,n) = \mu_n$ for all $n = 1,\dots,N$ if and only if $\{\lambda_n\}_{n=1}^N$ *majorizes* $\{\mu_n\}_{n=1}^N$, that is, precisely when

$$\sum_{n=1}^N \mu_n = \sum_{n=1}^N \lambda_n, \qquad \sum_{n=1}^j \mu_n \le \sum_{n=1}^j \lambda_n, \quad \forall j = 1,\dots,N, \tag{1}$$

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denoted $\{\mu_n\}_{n=1}^N \preceq \{\lambda_n\}_{n=1}^N$. The first part of (1) is simply a trace condition: the sum of the diagonal entries of $\mathbf{G}$ must equal the sum of its eigenvalues. The second part of (1) is less intuitive. To understand it better, it helps to have some basic concepts from finite frame theory.

For any finite sequence of vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{F}^M$, the corresponding *synthesis operator* is the $M \times N$ matrix whose $n$th column is $\boldsymbol{\varphi}_n$, namely $\boldsymbol{\Phi} : \mathbb{F}^N \to \mathbb{F}^M$, $\boldsymbol{\Phi}\mathbf{y} := \sum_{n=1}^N \mathbf{y}(n)\boldsymbol{\varphi}_n$. Its $N \times M$ adjoint is the *analysis operator* $\boldsymbol{\Phi}^* : \mathbb{F}^M \to \mathbb{F}^N$, $(\boldsymbol{\Phi}^*\mathbf{x})(n) := \langle \boldsymbol{\varphi}_n, \mathbf{x}\rangle$. The vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ are a *finite frame* for $\mathbb{F}^M$ if they span $\mathbb{F}^M$, which is equivalent to having their $M \times M$ *frame operator* $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{n=1}^N \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*$ be invertible. Here, $\boldsymbol{\varphi}_n^*$ is $1 \times M$ adjoint of the $M \times 1$ column vector $\boldsymbol{\varphi}_n$, namely the linear operator $\boldsymbol{\varphi}_n^*\mathbf{x} = \langle \boldsymbol{\varphi}_n, \mathbf{x}\rangle$. The least and greatest eigenvalues $\alpha$ and $\beta$ of $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ are called the *lower* and *upper frame bounds* of $\{\boldsymbol{\varphi}_n\}_{n=1}^N$, and their ratio $\beta/\alpha$ is the *condition number* of $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$. Inspired by applications involving additive noise, finite frame theorists often seek frames that are as well-conditioned as possible, the ideal case being *tight frames* in which $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \alpha\mathbf{I}$ for some $\alpha > 0$. They also care about the lengths of the frame vectors, often requiring that $\|\boldsymbol{\varphi}_n\|^2 = \mu_n$ for some prescribed sequence $\{\mu_n\}_{n=1}^N$. These lengths weight the summands of the linear-least-squares objective function $\|\boldsymbol{\Phi}^*\mathbf{x} - \mathbf{y}\|^2 = \sum_{n=1}^N |\langle \boldsymbol{\varphi}_n, \mathbf{x}\rangle - \mathbf{y}(n)|^2$, and adjusting them is closely related to the linear-algebraic concept of *preconditioning*. That is, we often want to control both the spectrum of the frame operator as well as the lengths of the frame vectors. For example, much attention has been paid to finite tight frames whose vectors are unit norm [2,5,14,15].

In this context, the reason we care about the Schur–Horn theorem is that it provides a simple characterization of when there exists a finite frame whose frame operator has a given spectrum and whose frame vectors have given lengths. To elaborate, the earliest reference which briefly mentions the Schur–Horn theorem in the context of finite frames seems to be [26], which stems from even earlier, closely related work on synchronous CMDA systems [27,28]. An in-depth analysis of the connection between frame theory and the Schur–Horn theorem is given in [1]. There as here, the main idea is to apply the Schur–Horn theorem to the *Gram matrix* of a given sequence of vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$, namely the $N \times N$ matrix $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ whose $(n, n')$th entry is $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})(n, n') = \langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'}\rangle$. Indeed, suppose there exists $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{F}^M$ whose frame operator $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ has spectrum $\{\lambda_m\}_{m=1}^M$ and whose frame vectors have squared-norms $\|\boldsymbol{\varphi}_n\|^2 = \mu_n$ for all $n = 1, \ldots, N$. The diagonal entries of $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ are $\{(\boldsymbol{\Phi}^*\boldsymbol{\Phi})(n, n)\}_{n=1}^N = \{\|\boldsymbol{\varphi}_n\|^2\}_{n=1}^N = \{\mu_n\}_{n=1}^N$ which, by reordering the frame vectors if necessary, we can assume are nonincreasing. Meanwhile, the spectra of the Gram matrix $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ and the frame operator $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ are zero-padded versions of each other. Since adjoining vectors of squared-length $\mu_n = 0$ to a sequence $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ does not change its $M \times M$ frame operator $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ we further assume without loss of generality that $M \leq N$, implying that the spectrum of $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ is $\{\lambda_m\}_{m=1}^M$ appended with $N - M$ zeros. Applying the Schur–Horn theorem to $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ then implies that $\{\lambda_m\}_{m=1}^M \cup \{0\}_{m=M+1}^N$ necessarily majorizes $\{\mu_n\}_{n=1}^N$, with (1) reducing to

$$\sum_{n=1}^N \mu_n = \sum_{m=1}^M \lambda_m, \qquad \sum_{n=1}^j \mu_n \leq \sum_{m=1}^j \lambda_m, \quad \forall j = 1, \ldots, M. \tag{2}$$

Conversely, for any $M \leq N$ and any nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ that satisfy (2), the Schur–Horn theorem also implies that there exists a positive semidefinite matrix with spectrum $\{\lambda_m\}_{m=1}^M \cup \{0\}_{m=M+1}^N$ and with diagonal entries $\{\mu_n\}_{n=1}^N$. Since the rank of $\mathbf{G}$ is at most $M$, taking the singular value decomposition of $\mathbf{G}$ allows it to be written as $\mathbf{G} = \boldsymbol{\Phi}^*\boldsymbol{\Phi}$ where $\boldsymbol{\Phi} \in \mathbb{F}^{M \times N}$ has singular values $\{\lambda_m^{1/2}\}_{m=1}^M$. Letting $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ denote the columns of this matrix $\boldsymbol{\Phi}$, we see that there exists $N$ vectors in $\mathbb{F}^M$ whose frame operator $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ has spectrum $\{\lambda_m\}_{m=1}^M$ and where $\|\boldsymbol{\varphi}_n\|^2 = \mu_n$ for all $n = 1, \ldots, N$.

In summary, for any $M \leq N$ and any nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$, the Schur–Horn theorem gives that there exists $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{F}^M$ where $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ has spectrum $\{\lambda_m\}_{m=1}^M$ and where $\|\boldsymbol{\varphi}_n\|^2 = \mu_n$ for all $n$ if and only if (2) holds. Note that in the $M = N$ case, this statement reduces the classical Schur–Horn theorem and as such, is an equivalent formulation of it. This equivalence allows the Schur–Horn and finite frame theory communities to contribute to each other. For example, the Schur–Horn