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Semi-continuous and discrete wavelet frames on n-dimensional spheres

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The paper shows that under some mild conditions n-dimensional spherical wavelets derived from approximate identities build semi-continuous frames. Moreover, for sufficiently dense grids Poisson wavelets on n-dimensional spheres constitute a discrete frame. In the proof we only use the localization properties of the reproducing kernel and its gradient.

1. Introduction

In the recent years, an interest on n-dimensional spherical wavelet transform has been growing. Besides discrete approaches [29,24] there are several continuous constructions [5] (being a generalization to n dimensions of spherical wavelets introduced in [4]), [13,14]. For an efficient usage of a continuous wavelet transform, a discretization algorithm is needed. Frames have been constructed for 2-dimensional spherical wavelets derived in [4], cf. [1,9], however, the phase-space discretization is performed on an equiangular grid, a solution that can hardly be applied in a higher dimension.

In this paper, we generalize the results obtained in [27] for 2-dimensional spherical wavelets. The construction of semi-continuous frames is similar to that in [1,9] for the two-dimensional sphere. As a next step, for each scale we perform a discretization of the spherical parameter such that the sampling points are quite uniformly distributed over the sphere. Finally, the sampling point positions are perturbed in such a way

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that the density of the resulting grid is controlled with respect to the scale and space parameter simultaneously. If the density is big enough, the discrete set of wavelets is a frame for $\mathcal{L}^2(\mathcal{S}^n)$. The constraints on the wavelets are some estimations on their reproducing kernel and its gradient, which are satisfied by Poisson multipole wavelets.

After the present research had been completed, the author learnt about a similar frame construction for Mexican needlets [19]. Therefore, the present discussion focuses on the differences and similarities in the end of the paper.

The paper is organized as follows: In Section 2 after a recapitulation of basic facts about the n-dimensional wavelet transform derived from approximate identities, and particularly Poisson multipole wavelets, we recall some information about frames in Hilbert spaces. Section 3 contains a discussion of a condition for semi-continuous frames. It is shown, that many popular wavelet families constitute semi-continuous frames. The main theorem of this paper about the phase-space discretization is to be found in Section 4, and a perturbation of this result to fully irregular frames controlled only by hyperbolic density is the topic of Section 5. It is shown in Section 6 that both discretization results apply to Poisson multipole wavelets. A comparison with the frame construction for Mexican needlets is presented in Section 7.

2. Preliminaries

By $\mathcal{S}^n$ we denote the n-dimensional unit sphere in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the rotation-invariant measure $d\sigma$ normalized such that

$$\Sigma_n = \int_{\mathcal{S}^n} d\sigma = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)},$$

where λ and n are related by

$$\lambda = \frac{n-1}{2}.$$

The surface element $d\sigma$ is explicitly given by

$$d\sigma = \sin^{n-1}\theta_1\, \sin^{n-2}\theta_2 \ldots \sin\theta_{n-1} d\theta_1\, d\theta_2 \ldots d\theta_{n-1} d\varphi,$$

where $(\theta_1, \theta_2, \ldots, \theta_{n-1}, \varphi) \in [0,\pi]^{n-1} \times [0, 2\pi)$ are spherical coordinates satisfying

$$\begin{aligned}
x_1 &= \cos\theta_1, \\
x_2 &= \sin\theta_1 \cos\theta_2, \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\ldots \\
x_{n-1} &= \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \cos\theta_{n-1}, \\
x_n &= \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \sin\theta_{n-1} \cos\varphi, \\
x_{n+1} &= \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \sin\theta_{n-1} \sin\varphi.
\end{aligned}$$

$\langle x, y\rangle$ or $x \cdot y$ stand for the scalar product of vectors with origin in O and endpoints on the sphere. As long as it does not lead to misunderstandings, we identify these vectors with points on the sphere. By $\angle(x,y)$ we denote the geodesic distance between two points $x, y \in \mathcal{S}^n$,

$$\angle(x,y) := \arccos(x,y).$$