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Characterization and analysis of edges in piecewise smooth functions

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ABSTRACT

The analysis and detection of edges is a central problem in applied mathematics and image processing.

A number of results in recent years have shown that directional multiscale methods such as continuous curvelet and shearlet transforms offer a powerful theoretical framework to capture the geometry of edge singularities, going far beyond the capabilities of the conventional wavelet transform. The continuous shearlet transform, in particular, provides a precise geometric characterization of edges in piecewise constant functions in $\mathbb{R}^2$ and $\mathbb{R}^3$, including corner points. However, a question has been raised frequently: What happens if the function is piecewise smooth and not just piecewise constant? Clearly, a piecewise smooth function is a much more realistic model of images with edges.

In this paper, we extend the characterization results previously known and show that, also in the case of piecewise smooth functions, the continuous shearlet transform can detect the location and orientation of edge points, including corner points, through its asymptotic decay at fine scales. The new proof introduces innovative technical constructions to deal with the more challenging problem. The new results set the theoretical groundwork for the application of the shearlet framework to a wider class of problems from image processing.

1. Introduction

Multiscale methods and wavelets have been frequently associated with the analysis of singularities of functions and distributions and the application of the wavelet transform to edge detection goes back to the origins of the wavelet literature [10]. In fact, it is not difficult to show that, if f is a function that is smooth apart from a discontinuity at a point p_0, then the continuous wavelet transform of f, denoted by $\mathcal{W}f(a,p)$, decays rapidly at asymptotically finer scales (as a approaches 0) unless p is near p_0 [17,28]. More generally,

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one can show that the continuous wavelet transform resolves the *singular support* of f, that is, the set of points where f is not regular, and can be applied to measure the pointwise regularity of functions [18,19].

However, the conventional wavelet approach is unable to provide additional information about the *geometry* of the set of singularities of f. In many applications, including propagation of singularities in PDEs and image processing, it is useful not only to identify the locations of the singular points, but also their geometrical properties, such as the orientation and curvature of an edge.

In the mathematical literature, the idea of using generalized wavelet-like transforms to perform a more sophisticated microlocal analysis of singularities can be traced back to Bros and Iagolnitzer [1] and Cordoba and Fefferman [3], who both defined transforms with implicitly a kind of anisotropic scaling. It was later shown that the Fourier–Bros–Iagolnitzer (FBI) transform is able to resolve not only the singular support but also the wavefront set of a distribution [8]. During the last decade, following the introduction of a new generation of directional multiscale systems in the wavelet literature, most notably the curvelet [2] and shearlet systems [21], it was shown that it is possible to define "generalized" wavelet transforms able to resolve the wavefront set of distributions [9,21]. Among such transforms, the continuous shearlet transform offers an especially appealing mathematical framework, due to a simple construction, derived from the general setting of affine systems, and its high directional sensitivity obtained through the action of anisotropic dilations and shear transformations. One major advantage of this approach is that, thanks to the affine structure, there is a rather direct procedure to derive discrete versions of the shearlet transform and transfer the theoretical properties of the continuous transform to its discrete counterparts [7,23].

In particular, it was shown by the authors and their collaborators that the continuous shearlet transform provides a precise *geometric* characterization of edge singularities for a large class of multidimensional functions and distributions [12–15], going far beyond the capabilities of wavelets and other conventional multiscale methods. For example, let us consider piecewise constant functions of the form $h = \sum_{i=1}^{N} c_i \chi_{S_i}$, where, for each i, c_i is a constant and S_i is a compact subset of $\mathbb{R}^2$. Such functions provide a very simple model of images with edges where the edge detection problem consists in identifying the boundary curves ∂S_i of the sets S_i. The continuous shearlet transform was found to be remarkably effective for this task, since it precisely characterizes the *location* and *orientation* of piecewise regular boundary curves ∂S_i, including possibly corner points.[1] Similar results were found in the three-dimensional setting [13,14]. We also mention a very recent version of these results using compactly supported shearlet generators by Kutyniok and Petersen [22], which includes uniform estimates.

The theoretical results available so far, however, are restricted for the most part to a very simplified model of images consisting of piecewise constant functions (cf. [11] for a recent survey about these theoretical results). To go beyond this limitation, in this paper, we consider a much more realistic model of images with edges that is not limited to characteristic functions of sets, but includes general smooth density functions. The geometric analysis and detection of edges in this situation is significantly more challenging and cannot be handled using the techniques and arguments developed in the previous studies. In order to illustrate these challenges and describe the main original contributions of this work, let us start by setting our notation and recalling the results currently known (in dimensions $n = 2$).

In this paper, we consider functions of the form

$$h(x) = \sum_{i=1}^{N} f_i(x)\, \chi_{S_i}(x), \tag{1.1}$$

where, for each i, f_i is a C^∞ (non-trivial) function and S_i is a compact region in $\mathbb{R}^2$ whose boundary, denoted by ∂S_i, is a simple, *piecewise smooth* curve, of finite length. To simplify our arguments, we assume

[1] Note that corner points and junctions frequently provide the most conspicuous and useful features for many algorithms of edge analysis and feature extraction (e.g. [25,33]).