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Fast proximity-gradient algorithms for structured convex optimization problems ☆

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ABSTRACT

We introduce in this paper fixed-point proximity-gradient algorithms for solving a class of structured convex optimization problems arising from image restoration. The objective function of such optimization problems is the sum of three convex functions. We study in this paper the scenario where one of the convex functions involved is differentiable with a Lipschitz continuous gradient and another convex function is composed by an affine transformation. We first characterize the solutions of the optimization problem as fixed-points of a mapping defined in terms of the gradient of the differentiable function and the proximity operators of the other two functions. Then, a fixed-point proximity-gradient iterative scheme is developed based on the fixed-point equation which characterizes the solutions. We establish the convergence of the proposed iterative scheme by the notion of averaged nonexpansive operators. Moreover, we obtain that in general the proposed iterative scheme has $O(\frac{1}{k})$ convergence rate in the ergodic sense and the sense of partial primal–dual gap. Under stronger assumptions on the convex functions involved the proposed iterative scheme will converge linearly. We in particular derive fixed-point proximity-gradient algorithms from the proposed iterative scheme. The quasi-Newton and the overrelaxation strategies are designed to accelerate the algorithms. Numerical experiments for the computerized tomography reconstruction problem demonstrate that the proposed algorithms perform favorably and the quasi-Newton as well as the overrelaxation strategies significantly accelerate the convergence of the algorithms.

1. Introduction

In this paper, we consider solving a class of convex optimization problems which minimizes the sum of three convex functions. We are interested in the scenario where one of the convex functions involved is differ-

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entiable with a Lipschitz continuous gradient and another function is composed by an affine transformation. This problem has important applications in image processing. For example, the total variation based image denoising model [24], sparse representation based image restoration [5,9,10,15], and the compressive sensing problem [7] can all be considered as special cases of this general setting.

In recent years, a number of efficient algorithms have been developed for convex problems which minimize the sum of two convex functions. The proximal forward–backward algorithm [1] and the Nesterov's first-order explicit schemes [3,22] are applicable when one convex function involved is differentiable with a Lipschitz continuous gradient. The split Bregman iteration [6,17] and the primal dual extrapolation algorithms [8,16] are very suitable if one convex function is composed by an affine transform. The primal–dual fixed-point algorithm [11] is specially designed for the case when one convex function is composed by an affine transformation and the other one is differentiable with a Lipschitz continuous gradient. The convex problems considered in this paper are different from the problems which minimize the sum of two convex functions. Although some of the algorithms mentioned above can be applied to the convex problems by some modification, we aim at developing fast algorithms based on properties of the three functions involved in the optimization problem.

Recently, there were a few research works regarding the extension of ADMM to the three-block case. These three-block ADMMs can be adapted for solving the considered three objective problem. It is worth noting that a new three operator splitting algorithm (3OSA) [14] was proposed very recently for solving this type of problem. The 3OSA can be also viewed as a variant of three-block ADMM. Our proposed algorithms differ from the ADMM-type algorithms in the iterative procedure. As we were finishing this paper, we found the algorithm proposed in [12] was very similar to our proposed algorithms. However, there are several differences between the two works. First, the algorithm proposed in [12] is actually a special case of our proposed algorithms when two precondition matrices in our algorithms are fixed as quantity matrices. Second, the mathematical tools used for analyzing the convergence of the algorithms are different and we analyze the convergence rate of the proposed algorithm while [12] does not. Third, we design two strategies which efficiently accelerate the convergence of the proposed algorithms. Finally, the paper [12] does not show any simulations while we apply our proposed algorithms to image restoration and present two efficient strategies to accelerate the algorithms.

As shown in one of our previous paper [19], the notion of proximity operators provides us a useful tool for the algorithmic development. When the proximity operators of the convex functions involved in the objective function have closed forms, the resulting algorithms are computationally efficient [20]. Particularly, the proximity operators of the non-differentiable convex functions that we encounter in the area of image restoration always have closed forms and thus can be computed fast. However, computing the proximity operator of the differentiable functions arising in image restoration may take much time. For instance, the fidelity term of the computerized tomography reconstruction model (shown in Section 6) is a quadratic term composed with a matrix. In order to implement its proximity operator, we have to solve a large scale linear system, costing much time. In contrast, computing the gradient of this fidelity term is much more efficient. Therefore, as the computation of the proximity operator of the differentiable function is time consuming, we resort to its gradient instead to develop fast algorithms.

In this paper, we use the gradient of the differentiable convex function of the objective function and the proximity operators of the other two convex functions to develop algorithms. We first transfer solving the optimization problem considered in this paper to searching for fixed-points of an operator, which is related to the gradient of the differentiable function of the objective function and the proximity operators of the other two convex functions. Then, by transforming the original fixed-point equations, we develop a fixed-point iterative scheme. Finally, we prove the convergence of the proposed iterative scheme with the help of the notion of averaged nonexpansive operators. We also design four fixed-point proximity-gradient algorithms and establish their convergence results based on the proposed iterative scheme. We point out that for some special convex problems, the proposed algorithms will reduce to the proximal forward backward method