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Directional tensor product complex tight framelets with low redundancy ☆

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ABSTRACT

Having the advantages of redundancy and flexibility, various types of tight frames have already shown impressive performance in applications such as image and video processing. For example, the undecimated wavelet transform, which is a particular case of tight frames, is known to have good performance for the denoising problem. Empirically, it is widely known that higher redundancy rate of a tight frame often leads to better performance in applications. The wavelet/framelet transform is often implemented in an undecimated fashion for the purpose of better performance in practice. Though high redundancy rate of a tight frame can improve performance in applications, as the dimension increases, it also makes the computational cost skyrocket and the storage of frame coefficients increase exponentially. This seriously restricts the usefulness of such tight frames for problems in moderately high dimensions such as video processing in dimension three. Inspired by the directional tensor product complex tight framelets TP-$\mathbb{C}$TF$_m$ with $m \geq 3$ in [15,20] and their impressive performance for image processing in [20,33], in this paper we introduce directional tensor product complex tight framelets TP-$\mathbb{C}$TF$^{\downarrow}_m$ (called reduced TP-$\mathbb{C}$TF$_m$) with low redundancy. Such TP-$\mathbb{C}$TF$^{\downarrow}_m$ are particular examples of tight framelet filter banks with mixed sampling factors. In particular, we shall develop a directional tensor product complex tight framelet TP-$\mathbb{C}$TF$^{\downarrow}_6$ such that it performs nearly as well as the original TP-$\mathbb{C}$TF$_6$ in [20] for image/video denoising/inpainting but it has significantly lower redundancy rates than TP-$\mathbb{C}$TF$_6$ in every dimension. The TP-$\mathbb{C}$TF$^{\downarrow}_6$ in d dimensions not only offers good directionality as the original TP-$\mathbb{C}$TF$_6$ does but also has the low redundancy rate $\frac{3^d-1}{2^d-1}$ (e.g., the redundancy rates are $2, 2\frac{2}{3}, 3\frac{5}{7}, 5\frac{1}{3}$ and $7\frac{25}{31}$ for dimension $d = 1, \dots, 5$, respectively), in comparison with the redundancy rate $2^d \times \frac{3^d-1}{2^d-1}$ of TP-$\mathbb{C}$TF$_6$ in dimension d. Moreover, our numerical experiments on image/video denoising and inpainting show that the performance using our proposed TP-$\mathbb{C}$TF$^{\downarrow}_6$ is often comparable with or

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sometimes better than several state-of-the-art frame-based methods which have much higher redundancy rates than that of TP-$\mathbb{C}\mathrm{TF}_6^{\downarrow}$.

1. Introduction and motivations

Though wavelets have many useful applications, they have several shortcomings in dealing with multi-dimensional problems. For example, tensor product real-valued wavelets are known for lack of the desired properties of translation invariance and directionality [6,23,31]. There are a lot of papers in the current literature to improve the performance of classical tensor product (i.e., separable) real-valued wavelets by remedying these two shortcomings. In one direction, translation invariance of wavelets can be improved by using wavelet frames instead of orthonormal wavelets (see [6,7,12,14,16–18,30,31] and many references therein). For example, the undecimated wavelet transform [6] using Daubechies orthonormal wavelets has been known to be effective for the denoising problem. In fact, such an undecimated wavelet transform employs a particular case of tight frames with high redundancy. A countable set $\{h_k\}_{k\in\Lambda}$ of elements in a Hilbert space $\mathcal{H}$ equipped with an inner product $\langle\cdot,\cdot\rangle$ is called *a frame* if there exist positive constants C_1 and C_2 such that

$$C_1\langle h,h\rangle \leqslant \sum_{k\in\Lambda} |\langle h,h_k\rangle|^2 \leqslant C_2\langle h,h\rangle, \qquad \forall\, h\in\mathcal{H}.$$

In particular, it is called a (normalized) *tight frame* if $C_1 = C_2 = 1$. If $\mathcal{H}$ is a finite dimensional space with dimension d, then the *redundancy rate* of a frame $\{h_k\}_{k\in\Lambda}$ is naturally defined to be $\frac{\#\Lambda}{d}$, where $\#\Lambda$ is the cardinality of the index set Λ. Note that an orthonormal basis in $\mathcal{H}$ is a particular tight frame with the redundancy rate one. Comparing with an orthonormal basis, a (tight) frame is more general and has redundancy by allowing more elements into its system. The added redundancy of a tight frame not only improves the property of translation invariance but also makes the design of a tight frame more flexible (see [6,7,12,14,16–18,21,30] and references therein). In the other direction, many papers in the literature have been studying directional representation systems, to only mention a few here, curvelets in [1,2,34], contourlets in [8], shearlets in [10,11,21,23–25,27,28] and many references therein, surfacelets in [29], dual tree complex wavelet transform in [22,31,32], complex tight framelets in [14,15,17,18,20], plus many other directional representation systems. To improve directionality of tensor product real-valued wavelets, due to the requirement of the additional angular resolution for a directional representation system, it is almost unavoidable to employ either a tight frame or a frame instead of an orthonormal basis by allowing redundancy. In fact, to our best knowledge, all currently known representation systems, having either directionality and/or (near) translation invariance, employ either a frame or a tight frame with various degrees of redundancy. The directional tensor product complex tight framelets in [15,20] and their reduced versions with low redundancy in this paper are different in nature from many known directional representation systems such as curvelets and shearlets. This issue will be addressed and explained in details in Section 3.2.

In the following, let us introduce a fast framelet transform and explain by what we mean the redundancy rate of a transform or a system. To this end, let us recall the definition of a tight framelet filter bank. For $u = \{u(k)\}_{k\in\mathbb{Z}^d} \in l_1(\mathbb{Z}^d)$, we define the *Fourier series* (or *symbol*) $\widehat{u}$ of the sequence u to be $\widehat{u}(\xi) := \sum_{k\in\mathbb{Z}^d} u(k)e^{-ik\cdot\xi}$, $\xi\in\mathbb{R}^d$. Note that $\widehat{u}$ is a $2\pi\mathbb{Z}^d$-periodic function satisfying $\widehat{u}(\xi+2\pi k) = \widehat{u}(\xi)$ for all $k\in\mathbb{Z}^d$. For $a, b_1, \ldots, b_s \in l_1(\mathbb{Z}^d)$, we say that $\{a; b_1, \ldots, b_s\}$ is a (d-dimensional dyadic) *tight framelet filter bank* if