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Sampling and Galerkin reconstruction in reproducing kernel spaces

Cheng Cheng [a], Yingchun Jiang [b,*], Qiyu Sun [a]

[a] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
[b] School of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin, Guangxi 541004, China

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ABSTRACT

In this paper, we introduce a fidelity measure depending on a given sampling scheme and we propose a Galerkin method in Banach space setting for signal reconstruction. We show that the proposed Galerkin method provides a quasi-optimal approximation, and the corresponding Galerkin equations could be solved by an iterative approximation-projection algorithm in a reproducing kernel subspace of L^p. Also we present detailed analysis and numerical simulations of the Galerkin method for reconstructing signals with finite rate of innovation.

1. Introduction

Digital processing of signals f may start from sampling on a discrete set Γ,

$$f \longmapsto \big(f(\gamma_n)\big)_{\gamma_n \in \Gamma} \tag{1.1}$$

[5,32,42,43]. The celebrated Whittaker–Shannon–Kotelnikov sampling theorem states that a bandlimited signal can be recovered from its samples taken at a rate greater than twice the bandwidth [32,45]. In

* Corresponding author.
E-mail addresses: cheng.cheng@knights.ucf.edu (C. Cheng), guilinjiang@126.com (Y. Jiang), qiyu.sun@ucf.edu (Q. Sun).

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the last two decades, that paradigm for bandlimited signals has been extended to represent signals in a shift-invariant space [5,7,42], signals with finite rate of innovation [13,28,31,34,36,43], and signals in a reproducing kernel space [11,19,24,29,30].

A fundamental problem in sampling theory is how to obtain a good approximation of the signal f when only the noisy sampling data $\big(f(\gamma_n)+\epsilon(\gamma_n)\big)_{\gamma_n\in\Gamma}$ is available [3,5,34,42]. The above problem is well studied and many algorithms, such as the frame algorithm and the approximation-projection algorithm, have been proposed [4,12,14,17,29,34,38]. In this paper, we introduce a Galerkin method for signal reconstruction and we propose a fast and stable algorithm to solve the corresponding Galerkin equations.

A conventional way to reconstruct signals f in a linear space V from their sampling data is to solve a minimization problem

$$Rf := \operatorname*{argmin}_{h\in V} \|h-f\|, \tag{1.2}$$

where the fidelity measure $\|h-f\|$ depends only on the sampling data of $h-f$ on Γ. Typical examples of fidelity measures in the bandlimited setting are weighted sampling energy $\sum_{\gamma_n\in\Gamma} w_n|f(\gamma_n)-h(\gamma_n)|^2$ and weighted pre-reconstruction energy $\|\sum_{\gamma_n\in\Gamma} w_n(f(\gamma_n)-h(\gamma_n))\mathrm{sinc}(\cdot-\gamma_n)\|_2$, where w_n are positive weights appropriately selected.

The fidelity of perceptual signals, such as acoustic and visual signals, might not be well measured by some weighted square errors [10,44]. Alternatives of fidelity measures are weighted sampling error $\big(\sum_{\gamma_n\in\Gamma} w_n|f(\gamma_n)-h(\gamma_n)|^p\big)^{\frac{1}{p}}$ and weighted pre-reconstruction error $\|\sum_{\gamma_n\in\Gamma} w_n(f(\gamma_n)-h(\gamma_n))K(\cdot,\gamma_n)\|_p$, $1\le p<\infty$, for signals in a reproducing kernel subspace of $L^p := L^p(\mathbf{R}^d)$ with kernel K. In this paper, we introduce a general fidelity measurement associated with a linear operator S on a Banach space V, that depends on the sampling scheme (1.1). Then the minimization problem (1.2) becomes

$$Rf := \operatorname*{argmin}_{h\in V} \|Sh-Sf\|_V. \tag{1.3}$$

The operator S in the above minimization problem can be selected as

$$Sf := \sum_{\gamma_n\in\Gamma} w_n f(\gamma_n)\mathrm{sinc}(\cdot-\gamma_n)$$

for the bandlimited setting, and

$$Sf := \sum_{\gamma_n\in\Gamma} w_n f(\gamma_n)K(\cdot,\gamma_n)$$

for the reproducing kernel space setting.

The nonlinear minimization problem (1.3) does not give a tractable signal reconstruction. Observe that

$$\|Sh-Sf\|_V = \sup_{\|g\|_{V^*}=1, g\in V^*} |\langle Sh-Sf, g\rangle|,$$

where $\langle\cdot,\cdot\rangle$ is the standard dual product between elements in V and its dual V^*. So we propose the following linear approach

$$\langle Sh, g\rangle = \langle Sf, g\rangle \quad \text{for all } g\in\tilde U, \tag{1.4}$$

where $\tilde U \subset V^*$ is a (finite-dimensional) trial space. Clearly, the solution of the Galerkin equations (1.4) with $\tilde U = V^*$ is also a solution of the minimization problem (1.3).